4-REGULAR INTEGRAL GRAPHS

D. Cvetković, S.K. Simić, D. Stevanović

Possible spectra of 4-regular integral graphs are determined. Some constructions and a list of 65 known connected 4-regular integral graphs are given.

1. INTRODUCTION

A graph is called integral if all its eigenvalues (of the adjacency matrix) are integers. The quest for integral graphs was initiated by F. Harary and A. J. Schwenk [12]. All such connected cubic graphs were obtained by D. Cvetković and F. C. Bussemaker [6,3], and independently by A. J. Schwenk [15]. There are exactly thirteen connected cubic integral graphs. In fact, D. Cvetković [6] proved that the set of connected regular integral graphs of a fixed degree is finite. Similarly, the set of connected integral graphs with bounded vertex degrees is finite. Z. Radosavljević and S. Simić [16] determined all 13 connected nonregular nonbipartite integral graphs whose maximum degree equals four. The corresponding problem for bipartite graphs is not yet answered – see [16] for some details. Recently, D. Stevanović [17] determined all 24 connected 4-regular integral graphs avoiding ±3 in the spectrum. In this paper we are interested in connected 4-regular integral graphs.

The search for integral graphs becomes easier if we use the following product of graphs. Given two graphs $G$ and $H$, with vertex sets $V(G)$ and $V(H)$, their product $G \times H$ is the graph with the vertex set $V(G) \times V(H)$ in which two vertices $(x, a)$ and $(y, b)$ are adjacent if and only if $x$ is adjacent to $y$ in $G$ and $a$ is adjacent to $b$ in $H$. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $G$, and $\mu_1, \ldots, \mu_m$ are the eigenvalues of $H$, then the eigenvalues of $G \times H$ are $\lambda_i \mu_j$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$ (see [8]). If $G$ is connected, nonbipartite, 4-regular and integral, then the product $G \times K_2$ is connected, bipartite, 4-regular and integral, since the eigenvalues of $K_2$ are $1$ and $-1$. Therefore, in determining 4-regular integral graphs we can consider bipartite graphs only, and later extract such nonbipartite graphs $G$ from the decompositions of bipartite ones in the form $G \times K_2$. On the other hand, if $G$ is bipartite then $G \times K_2 = 2G$ and we cannot obtain new graphs by iterating the product with $K_2$.

In Section 2 we find all possible spectra of connected bipartite 4-regular integral graphs. In Section 3 we give some upper bounds on the number of vertices in $r$-regular bipartite graphs. Together with results announced in [19], it follows that, except for 5 exceptional spectra, each 4-regular bipartite integral graph has

1991 Mathematics Subject Classification: 05C50
at most 1260 vertices. In Section 4 we give the list of known 4-regular integral graphs. Section 5 contains concluding remarks.

2. SPECTRA OF 4-REGULAR BIPARTITE INTEGRAL GRAPHS

Suppose that $G$ is a 4-regular bipartite integral graph. Regular bipartite graphs have the same number of vertices in each part so that we may assume that they have $p = 2n$ vertices. Using superscripts to represent multiplicities, we shall write its spectrum in the form

$$[4, 3^x, 2^y, 1^z, 0^w, -1^x, -2^y, -3^z, -4].$$

Let further $q$ and $h$ denote the numbers of quadrilaterals and hexagons in $G$. It is well known that the sum of $k^{th}$ powers of the eigenvalues is just the number of closed walks of length $k$. Therefrom we get the following result.

**Lemma 1.** The parameters $n, x, y, z, w, q, h$ satisfy the Diophantine equations:

1. $\frac{1}{2} \sum \lambda_i^0 = 1 + x + y + z + w = n,$
2. $\frac{1}{2} \sum \lambda_i^2 = 16 + 9x + 4y + z = 4n,$
3. $\frac{1}{2} \sum \lambda_i^4 = 256 + 81x + 16y + z = 28n + 4q,$
4. $\frac{1}{2} \sum \lambda_i^6 = 4096 + 729x + 64y + z = 232n + 72q + 6h.$

Another useful lemma is due to A.J. Hoffman [13]. Let $G$ be a regular graph with distinct eigenvalues $\mu_1, \mu_2, \ldots, \mu_k$, and let $A$ be the adjacency matrix of $G$ ($J$ stands for all 1-matrix, and $I$ is the identity matrix).

**Lemma 2.** The adjacency matrix $A$ satisfies the equation

$$k \prod_{i=2}^k (r - \mu_i)J = p \prod_{i=2}^k (A - \mu_i I).$$

The search for possible spectra is divided in cases depending on the greatest integer less than 4 which is avoided in the spectrum of $G$ (see 1–5).

1. Case $x = 0$

By putting $x = 0$ in the equations (1)–(4) and then eliminating $n$ and $y$ from (1) and (2) we get $3z + 4w = 12$, and since $z, w \geq 0$, the only possibilities are $(z, w) = (0, 3)$ or $(z, w) = (4, 0).$
Subcase \((z, w) = (0, 3)\)

By substituting the values of \(x, z\) and \(w\) in (1–4) and putting \(y = n - 4\) (which follows from (1)) into (3) and (4) we get

\[
\begin{align*}
(6) \quad 3n + q &= 48, \\
(7) \quad 28n + 12q + h &= 640.
\end{align*}
\]

From (6) it follows that \(q = 3q_1\), and thus \(n + q_1 = 16\), which shows that \(n \leq 16\). Further, from (6) and (7) we get \(h = 8n + 64\). If \(y = 0\) then \(n = 4\) and the corresponding spectrum is shown in Table 1. If \(y > 0\) then Hoffman’s identity (5) reads \(384J = 2n(A^4 - 4A^2) + 8n(A^3 - 4A)\). Graph \(G\) is bipartite, and if we take vertices \(u, v\) from distinct parts (colours), then \((A^2)_{u,v} = 0\). So we have \(384 = 8n(A^3)_{u,v} - 4A_{u,v}\), wherefrom \(8n \mid 384\), i.e. \(n \mid 48\). Since \(n \leq 16\), the only possible values for \(n\) are 6, 8, 12, 16. The corresponding spectra are shown in Table 1. Here and in other tables of possible spectra the column “Label” indicates the existence of graphs with the corresponding spectrum by referring to the list of known 4-regular integral graphs in Section 4. If the graph with the corresponding spectrum does not exist, then this column contains symbol “-”. If all the graphs with the given spectrum are known, then the column “All” contains symbol “+”.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(x)</th>
<th>(y)</th>
<th>(z)</th>
<th>(w)</th>
<th>(q)</th>
<th>(h)</th>
<th>Label</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>36</td>
<td>96</td>
<td>(J_{8,1})</td>
<td>+</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>30</td>
<td>112</td>
<td>(J_{12,4})</td>
<td>+</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>3</td>
<td>24</td>
<td>128</td>
<td>(J_{16,1-2})</td>
<td>+</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>8</td>
<td>0</td>
<td>3</td>
<td>12</td>
<td>160</td>
<td>(J_{32,1-2})</td>
<td>+</td>
</tr>
<tr>
<td>16</td>
<td>0</td>
<td>12</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>192</td>
<td>(J_{32,1})</td>
<td>+</td>
</tr>
</tbody>
</table>

Table 1: The possible integral graph spectra with \((x, z, w) = (0, 0, 3)\).

Subcase \((z, w) = (4, 0)\)

By substituting the values of \(x, z\) and \(w\) in (1–4) and putting \(y = n - 5\) (which follows from (1)) into (3) and (4) we get

\[
\begin{align*}
(8) \quad 3n + q &= 45, \\
(9) \quad 28n + 12q + h &= 630.
\end{align*}
\]

From (8) it follows \(q = 3q_1\), and thus \(n + q_1 = 15\), which shows that \(n \leq 15\). Further, from (8) and (9) we get \(h = 8n + 90\). If \(y = 0\) then \(n = 5\) and the corresponding spectrum is shown in Table 2. If \(y > 0\) Hoffman’s identity (5) reads \(1440J = 2n(A^5 - 5A^3 + 4A) + 8n(A^4 - 5A^2 + 4)\), therefore \(n \mid 180\), and since \(n \leq 15\), the only possible values for \(n\) are 6, 9, 10, 12, 15. The corresponding spectra are shown in Table 2.
Table 2: The possible integral graph spectra with \((x, z, w) = (0, 4, 0)\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>(x)</th>
<th>(y)</th>
<th>(z)</th>
<th>(w)</th>
<th>(q)</th>
<th>(h)</th>
<th>Label</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>30</td>
<td>130</td>
<td>(J_{10,1})</td>
<td>+</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td>27</td>
<td>138</td>
<td>(J_{12,2})</td>
<td>+</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>18</td>
<td>162</td>
<td>(J_{18,1})</td>
<td>+</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>5</td>
<td>4</td>
<td>0</td>
<td>15</td>
<td>170</td>
<td>(J_{20,1})</td>
<td>+</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>7</td>
<td>4</td>
<td>0</td>
<td>9</td>
<td>186</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>0</td>
<td>10</td>
<td>4</td>
<td>0</td>
<td>200</td>
<td>210</td>
<td>(J_{30,1})</td>
<td>+</td>
</tr>
</tbody>
</table>

2. Case \(x > 0, y = 0\)

By putting \(y = 0\) in the equations (1)-(4) and subtracting (2) from (3) we obtain
\(240 + 72x = 24n + 4q\), from which \(q = 6q'\), and
\(3x = n + q' - 10\).

On the other hand, subtracting (1) from (2) yields
\(8x = 3n + w - 15\).

Eliminating \(x\) from (10) and (11) we obtain \(n = 8q' - 3w - 35\), and since \(x, z, h \geq 0\), we get the following conditions:

\(x = 3q' - w - 15 \geq 0\),
\(z = 5q' - 3w - 21 \geq 0\),
\(h = 210 - 16q' - 6w \geq 0\).

Hoffman’s identity (5) now reads
\(3360J = 2n(A^6 - 10A^4 + 9A^2) + 8n(A^5 - 10A^3 + 9A)\). Thus, we get \(n \mid 420\). By solving the system (12)-(14) for \(q'\) and \(w\), and eliminating the spectra not satisfying \(n \mid 420\), we get the possibilities shown in Table 3. Nonexistence of graphs with spectra indicated by “-” is shown in [19].

3. Case \(x > 0, y > 0, z = 0\)

By putting \(z = 0\) in the equations (1)-(4) and eliminating \(n\) from (1) and (2) we obtain \(4w = 12 + 5x\), so that \(x = 4x_1\). Substituting the value of \(n\) from (2) into (3) we get \(q = 36 + 18x_1 - 3y\), so that \(q = 3q_1\) and
\(q_1 = 12 + 6x_1 - y\).

Substituting the value of \(n\) from (2) and the value of \(q_1\) from (15) into (4) we get
\(\frac{1}{4}h = 48 - 39x_1 + 4y \geq 0\), and therefore
\(y \geq \frac{39}{4}x_1 - 12\).

\(^1\) This will be proven elsewhere.
From (15) and (16) we get

\begin{equation}
0 \leq q_1 \leq 24 - \frac{15}{4} x_1,
\end{equation}

which gives that \( x_1 \leq 6 \). Eliminating \( y \) from (2) and (15) yields \( n = 15x_1 - q_1 + 16 \), and from (17) it follows \( \frac{15}{4} x_1 - 8 \leq n \leq 15x_1 + 16 \). Hoffman’s identity (5) now reads \( 2688J = 2n(A^6 - 13A^3 + 36A) + 8n(A^5 - 13A^2 + 36A) \), and therefore \( n \mid 336 \). The corresponding spectra are shown in Table 4.

### Table 4: The possible integral graph spectra with \( x, y > 0, z = 0 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
<th>( w )</th>
<th>( q )</th>
<th>( h )</th>
<th>( \text{Label} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>14</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>8</td>
<td>51</td>
<td>26</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>4</td>
<td>3</td>
<td>0</td>
<td>8</td>
<td>45</td>
<td>42</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>4</td>
<td>8</td>
<td>0</td>
<td>8</td>
<td>30</td>
<td>82</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>4</td>
<td>11</td>
<td>0</td>
<td>8</td>
<td>21</td>
<td>106</td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>4</td>
<td>15</td>
<td>0</td>
<td>8</td>
<td>9</td>
<td>138</td>
<td></td>
</tr>
<tr>
<td>42</td>
<td>8</td>
<td>20</td>
<td>0</td>
<td>13</td>
<td>12</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>56</td>
<td>12</td>
<td>25</td>
<td>0</td>
<td>18</td>
<td>15</td>
<td>62</td>
<td></td>
</tr>
</tbody>
</table>

### 4. Case \( x > 0, y > 0, z > 0, w = 0 \)

Eliminating \( y \) and \( z \) from (1) and (2) we get \( y = n - \frac{8}{3} x - 5 \) and \( z = \frac{5}{3} x + 4 \), from which \( x = 3x_1 \). Eliminating then \( q \) and \( h \) from (3) and (4), we obtain \( q = -3n + 30x_1 + 45 \) and \( h = 8n - 80x_1 + 90 \). Since \( y \geq 1 \) and \( q, h \geq 0 \), we get

\[
\max \left\{ 1, \frac{n}{10} - \frac{3}{2} \right\} \leq x_1 \leq \min \left\{ \frac{n}{10} + \frac{9}{8}, \frac{n}{8} - \frac{3}{4} \right\}.
\]
Hoffman’s identity (5) now reads

$$10080 \cdot J = 2n(A^7 - 14A^5 + 49A^3 - 36A) + 8n(A^6 - 14A^4 + 49A^2 - 36I),$$

therefore $n \mid 1260$. The possible spectra are shown in Table 5. Nonexistence of graphs with spectra indicated by “−” is shown in [19].
5. Case \( x > 0, y > 0, z > 0, w > 0 \)

Eliminating \( z, w, q, h \) from (1)-(4), we get
\[
\begin{align*}
z &= -9x - 4y + 4n - 16, \\
w &= 8x + 3y - 3n + 15, \\
q &= 18x + 3y - 6n + 60, \\
h &= -96x - 26y + 34n - 40.
\end{align*}
\]

Hoffman’s identity (5) now reads
\[
40320 \cdot J = 2n(A^8 - 14A^6 + 49A^4 - 36A^2) + 8n(A^7 - 14A^5 + 49A^3 - 36A),
\]

therefore \( n \mid 5040 \). Since \( z, w \geq 1 \) and \( q, h \geq 0 \), we can easily (using a computer program) obtain the possible spectra. There are 1803 such spectra. In Subsection 3 we show that the upper bound for \( n \) is 3280, hence the case \( n = 5040 \) is impossible.

In [19] graph angles are exploited to show the nonexistence of graphs with some of these spectra, reducing the total number of potential spectra in this case to 1259.

It is also shown there that there are only 5 potential spectra with \( 630 < n \leq 2520 \). In Table 7 we have shown those with \( n \leq 20 \) and these 5 exceptional spectra. In Table 6 for each \( n \mid 5040, n \leq 630 \) we give the number \( SP \) of potential spectra.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( SP )</th>
<th>( n )</th>
<th>( SP )</th>
<th>( n )</th>
<th>( SP )</th>
<th>( n )</th>
<th>( SP )</th>
<th>( n )</th>
<th>( SP )</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>12</td>
<td>40</td>
<td>24</td>
<td>63</td>
<td>38</td>
<td>90</td>
<td>44</td>
<td>140</td>
<td>40</td>
</tr>
<tr>
<td>24</td>
<td>14</td>
<td>42</td>
<td>27</td>
<td>70</td>
<td>40</td>
<td>165</td>
<td>44</td>
<td>144</td>
<td>42</td>
</tr>
<tr>
<td>28</td>
<td>17</td>
<td>45</td>
<td>29</td>
<td>72</td>
<td>35</td>
<td>112</td>
<td>39</td>
<td>168</td>
<td>32</td>
</tr>
<tr>
<td>30</td>
<td>18</td>
<td>48</td>
<td>31</td>
<td>80</td>
<td>35</td>
<td>120</td>
<td>44</td>
<td>180</td>
<td>37</td>
</tr>
<tr>
<td>35</td>
<td>22</td>
<td>56</td>
<td>35</td>
<td>84</td>
<td>39</td>
<td>126</td>
<td>47</td>
<td>210</td>
<td>47</td>
</tr>
<tr>
<td>36</td>
<td>23</td>
<td>60</td>
<td>33</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6: The number of potential integral graph spectra with \( x, y, z, w > 0 \) and \( 21 \leq n \leq 630 \).

**3. UPPER BOUND ON THE NUMBER OF VERTEGES**

Here we give the upper bound on the number of vertices in regular bipartite graph, from which follows that there are no 4-regular bipartite integral graphs with \( n = 5040 \). This upper bound presents improvement in case of bipartite graphs upon previously known upper bound ([11]) for regular graphs of given diameter.

**Theorem 3.** Let \( p \) be the number of vertices of a connected \( r \)-regular bipartite graph \( G = (V, E) \) with radius \( R \). Then
\[
p \leq \frac{2(r - 1)^R - 2}{r - 2}.
\]
Table 7: The possible integral graph spectra with \(x, y, z, w > 0\) and \(n \leq 20\), and 5 exceptional spectra.

\[
\begin{array}{cccccc|c}
 n & x & y & z & w & q & h & \text{Label} \\
8 & 1 & 1 & 3 & 2 & 33 & 110 & I_{30,5} \\
9 & 1 & 2 & 3 & 2 & 30 & 118 & \\
10 & 1 & 3 & 3 & 2 & 27 & 126 & I_{34,3} \\
12 & 1 & 5 & 3 & 2 & 21 & 142 & I_{36,1} \\
12 & 2 & 6 & 1 & 30 & 124 & I_{36,2} \\
12 & 2 & 3 & 2 & 4 & 33 & 98 & I_{36,3} \\
12 & 3 & 1 & 1 & 6 & 45 & 54 & \\
14 & 1 & 7 & 3 & 2 & 15 & 158 & \\
14 & 2 & 4 & 6 & 1 & 24 & 140 & \\
14 & 2 & 5 & 2 & 4 & 27 & 114 & \\
14 & 3 & 2 & 5 & 3 & 36 & 96 & \\
14 & 3 & 3 & 1 & 6 & 39 & 70 & \\
15 & 1 & 8 & 3 & 2 & 12 & 166 & I_{40,2-3} \\
15 & 2 & 5 & 6 & 1 & 21 & 148 & \\
15 & 2 & 6 & 2 & 4 & 24 & 122 & \\
15 & 3 & 3 & 5 & 3 & 33 & 104 & \\
15 & 3 & 4 & 1 & 6 & 36 & 78 & \\
15 & 4 & 4 & 5 & 45 & 60 & \\
16 & 1 & 9 & 3 & 2 & 9 & 174 & \\
16 & 2 & 6 & 6 & 1 & 18 & 156 & \\
16 & 2 & 7 & 2 & 4 & 21 & 130 & \\
16 & 3 & 4 & 5 & 3 & 30 & 112 & \\
16 & 3 & 5 & 1 & 6 & 33 & 86 & \\
16 & 4 & 1 & 8 & 2 & 39 & 94 & \\
16 & 4 & 2 & 4 & 5 & 42 & 68 & \\
720 & 208 & 172 & 304 & 35 & 0 & 0 & I_{40,1-2} \\
840 & 244 & 196 & 364 & 35 & 0 & 0 & \\
1260 & 370 & 280 & 574 & 35 & 0 & 0 & \\
1680 & 496 & 364 & 784 & 35 & 0 & 0 & \\
2520 & 748 & 532 & 1204 & 35 & 0 & 0 & \\
\end{array}
\]

Proof. Let \(u\) be a vertex of \(G\) with eccentricity \(R\). Let \(N_k(u) = \{v \in V | d(u, v) = k\}\) be the \(k\)th neighborhood of \(u\), and \(d_k(u) = |N_k(u)|\) the number of vertices in it. Then \(d_0(u) = 1\) and \(d_1(u) = r\). When \(2 \leq k \leq R-1\), it holds \(d_k(u) \leq (r-1)d_{k-1}(u)\), since each vertex of \(N_{k-1}(u)\) may be adjacent to at most \(r-1\) vertices of \(N_k(u)\). Therefore by induction \(d_k(u) \leq r(r-1)^{k-1}\) for \(2 \leq k \leq R-1\). On the other hand, each vertex of \(N_R(u)\) is adjacent to \(r\) vertices of \(N_{R-1}(u)\) (since \(G\) is bipartite no two vertices of \(N_R(u)\) may be adjacent) so \(d_R(u) \leq r^{-1}d_{R-1}(u) \leq (r-1)^{R-1}\). Now \(p = \sum_{k=1}^{R} d_k(u) \leq \frac{2(r-1)^{R-1}}{r^{R-1}}\), and the theorem is proved. \(\square\)

As noted in [5], for the diameter \(D\) of a connected graph \(G\) the inequality \(D \leq s-1\) holds, where \(s\) is the number of distinct eigenvalues of \(G\). When \(G\) is
connected regular integral graph of degree \( r \), we get \( D \leq 2r \). From Theorem 3 then follows that connected bipartite 4-regular integral graph has diameter at most 8 and it may have at most \( p = 2n \leq 6560 \) vertices. But from the previous section it can be seen that the highest possible value for \( n \) not greater than 3280 is 2520. This bound cannot be lowered, but using graph angles it is shown in [19] that there are only 5 possible spectra with \( 650 < n \leq 2520 \).

4. A LIST OF KNOWN 4-REGULAR INTEGRAL GRAPHS

In this section we give a list of 65 connected 4-regular integral graphs. A part of these graphs are known in the literature. The others appear here for the first time and are derived from the known graphs by some construction procedures described below. We give to each of these graphs a label of the form \( I_{p,a} \) where \( p \) is the number of vertices, and \( a \) is the ordering number of graph in the group of graphs with the same number of vertices. For each graph we give its spectrum and a short description of the graph, sometimes with references to the literature. Graphs in the list are sorted by the number of vertices and then by the spectra.

Operations \( + \) and \( \times \) denote the usual graph sum and product. Operation \( \oplus \) denotes the strong sum: given two graphs \( G \) and \( H \), with vertex sets \( V(G) \) and \( V(H) \), their strong sum \( G \oplus H \) is the graph with the vertex set \( V(G) \times V(H) \) in which two vertices \((x,a)\) and \((y,b)\) are adjacent if and only if \( a \) is adjacent to \( b \) in \( H \) and \( x \) is adjacent to \( y \) in \( G \) or \( x = y \). \( L(G) \) and \( S(G) \) are the line graph and the subdivision graph (of \( G \)), respectively, while \( L_2(G) = L(S(G)) \).

Graphs \( G_1, \ldots, G_{13} \) are connected cubic integral graphs from \([15]\) \( (G_1, \ldots, G_8 \) are bipartite, while \( G_9, \ldots, G_{13} \) are not). In different order they are also given in \([6, 3]\).

In the following list we give their spectra.

\( G_1 = K_{3,3} : [3, 0^4, -3] \).
\( G_2 = K_2 + K_2 + K_2 = G_0 \times K_2 \) (the cube graph):
\[ [3, 1^3, -1^3, -3] \].
\( G_3 \) (Tutte’s 8-cage): \([3, 2^9, 0^{11}, -2^8, -3]\).
\( G_4 = G_{10} \times K_2 = G_{11} \times K_2 \) (the Desargues graph):
\[ [3, 2^4, 1^5, -1^5, -2^4, -3] \].
\( G_5 : [3, 2^4, 1^5, -1^5, -2^4, -3] \).
\( G_6 : [3, 2, 1^3, 0^3, -1^3, -2, -3] \).
\( G_7 = K_2 + C_5 = G_{12} \times K_2 \) (the 6-sided prism):
\[ [3, 2^2, 1, 0^4, -1, -2^2, -3] \].
\( G_8 = G_{13} \times K_2 : [3, 2^8, 1^3, 0^4, -1^3, -2^6, -3] \).
\( G_9 = K_4 : [3, -1^5] \).
\( G_{10} \) (the Petersen graph):
\[ [3, 1^5, -2^4] \).
\( G_{11} : [3, 2, 1^3, -1^3, -2^7] \).
\( G_{12} = K_2 + K_3 \) (the 3-sided prism):
\[ [3, 1, 0^3, -2^3] \).
\( G_{13} = L_2(K_4) : [3, 2^8, 0^3, -1^3, -2^3] \).
For each $G_i$ ($i = 1, \ldots, 13$) the graphs $L(G_i), L(G_i) \times K_2, G_i + K_2$ and $G_i \oplus K_2$ are connected 4-regular integral graphs. In the Table 8 we give their positions in the list of 4-regular integral graphs. If $G_i$ is bipartite, then $G_i + K_2 = G_i \oplus K_2$, so we do not double the column for these graphs.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$L(G_i)$</th>
<th>$L(G_i) \times K_2$</th>
<th>$G_i + K_2$</th>
<th>$G_i \oplus K_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$I_{5,2}$</td>
<td>$I_{18,1}$</td>
<td>$I_{12,2}$</td>
<td>$I_{8,2}$</td>
</tr>
<tr>
<td>2</td>
<td>$I_{12,7}$</td>
<td>$I_{24,2}$</td>
<td>$I_{16,1}$</td>
<td>$I_{29,4}$</td>
</tr>
<tr>
<td>3</td>
<td>$I_{14,1}$</td>
<td>$I_{8,1}$</td>
<td>$I_{30,3}$</td>
<td>$I_{29,7}$</td>
</tr>
<tr>
<td>4</td>
<td>$I_{30,4}$</td>
<td>$I_{6,1}$</td>
<td>$I_{48,1}$</td>
<td>$I_{12,5}$</td>
</tr>
<tr>
<td>5</td>
<td>$I_{30,5}$</td>
<td>$I_{6,1}$</td>
<td>$I_{48,1}$</td>
<td>$I_{12,5}$</td>
</tr>
<tr>
<td>6</td>
<td>$I_{15,3}$</td>
<td>$I_{23,2}$</td>
<td>$I_{28,5}$</td>
<td>$I_{24,5}$</td>
</tr>
<tr>
<td>7</td>
<td>$I_{18,5}$</td>
<td>$I_{36,1}$</td>
<td>$I_{24,3}$</td>
<td>$I_{24,5}$</td>
</tr>
<tr>
<td>8</td>
<td>$I_{36,4}$</td>
<td>$I_{72,1}$</td>
<td>$I_{48,1}$</td>
<td>$I_{24,5}$</td>
</tr>
</tbody>
</table>

Table 8: The positions of graphs $L(G_i), L(G_i) \times K_2, G_i + K_2$ and $G_i \oplus K_2$ in the list.

Let $G_i$ be nonbipartite, i.e. $i = 9, 10, \ldots, 13$. Graphs $(G_i + K_2) \times K_2$ and $(G_i \times K_2) + K_2$ are connected, 4-regular integral graphs which, in addition, are cospectral. However, we have the following proposition.

**Proposition 1.** Let $G$ be a nonbipartite graph. Graphs $(G + K_2) \times K_2$ and $(G \times K_2) + K_2$ are isomorphic.

**Proof.** For vertices of $(G + K_2) \times K_2$ and $(G \times K_2) + K_2$ we will shortly write $uab$ instead of $(u, a, b)$. Then $(u_0a_0b_0, u_1a_1b_1) \in E((G + K_2) \times K_2) \Leftrightarrow (u_0a_0b_0u_1a_1b_1) \in E(G + K_2)$ and $b_0 \neq b_1 \Rightarrow (u_0, u_1) \in E(G), a_0 = a_1, b_0 \neq b_1$ or $u_0 = u_1, a_0 \neq a_1, b_0 \neq b_1$; and $(u_0a_0b_0, u_1a_1b_1) \in E((G \times K_2) + K_2) \Leftrightarrow (u_0a_0u_1a_1b_1) \in E(G \times K_2), b_0 = b_1$ or $u_0a_0 = u_1a_1, b_0 \neq b_1 \Rightarrow (u_0, u_1) \in E(G), a_0 = a_1, b_0 = b_1$ or $u_0 = u_1, a_0 = a_1, b_0 \neq b_1$. Now it is not hard to see that the mapping $f: V((G + K_2) \times K_2) \mapsto V((G \times K_2) + K_2)$ given by $f(u, a, b) = (u, a + b, a)$, where addition is modulo 2, is actually the isomorphism of these graphs. □

Hence, all graphs $(G_i + K_2) \times K_2$ and $(G_i \times K_2) + K_2$ for $i = 9, 10, \ldots, 13$ are contained in the last row of Table 8.

Graphs $D_1, \ldots, D_8$ and $E_1, \ldots, E_8$ are connected, 4-regular and integral, as found in [17]; they are labelled (in the list) by $I_{8,1}, I_{18,1}, I_{12,4}, I_{12,2}, I_{16,1}, I_{16,2}, I_{24,1}, I_{24,2}, I_{32,1}, I_{18,3}, I_{18,3}, I_{20,1}, I_{20,2}, I_{20,3}, I_{20,1}$. Graphs $E_1, \ldots, E_8$ are labelled by $I_{5,1}, I_{6,1}, I_{8,2}, I_{12,6}, I_{12,7}, I_{4,4}, I_{8,2}, I_{15,2}$.

All graphs in the list up to 12 vertices have also been generated in [1] (see Fig. 21; note that $E_5$, i.e. the graph No 17, is not well drawn - see also [17]).
The list:

\[ I_{5,1} = E_1 = K_5 : [4, -1^4]. \]

\[ I_{6,1} = E_2 = L(K_4) = L(G_9) : [4, 0^3, -2^3], \]
also NEPS of \( C_3 \) and \( K_2 \) with basis \( \{(1, 1), (1, 0)\} \), cf. \([18]\).

\[ I_{7,1} = \overline{C_4 \cup C_5^2} : [1, 4, 0^3, -1^3, -3], \]
cf. graph 7402 in tables of \([10]\).

\[ I_{8,1} = D_1 = G_9 \oplus K_2 = K_{4,4} : [4, 0^6, -4], \]
cf. graph 8406 in tables of \([10]\).

\[ I_{8,2} = E_3 = G_9 + K_2 = \overline{K_6} : [4, 2, 0^3, -2^3], \] cf. graph 8401 in tables of \([10]\).

\[ I_{9,1} : [4, 1^3, 0^2, -2^3, -3], \] cf. graph 9414 in tables of \([10]\).

\[ I_{9,2} = E_7 = L(G_1) = C_3 + C_3 = C_3 \times C_3 : [4, 1^4, -2^4], \] cf. graph 9410 in tables of \([10]\).

\[ I_{9,3} : [4, 2, 1, 0^2, -1^2, -2, -3], \] cf. graph 9407 in tables of \([10]\).

\[ I_{9,4} = E_6 = L(G_{12}) : [4, 3, 2, 1^2, -1^2, -2^3], \]
cf. graph 9404 in tables of \([10]\).

\[ I_{10,1} = I_{5,1} \times K_3 \times D_2 = \overline{K_7} : [4, 1^4, -1^4, -4]. \]

\[ I_{12,1} : [4, 1^6, 0, -2^2, -3^2], \] NEPS of \( C_3, K_2 \) and \( K_2 \) with basis \( \{(1, 1, 0), (0, 0, 1), (0, 0, 1)\} \), [18].

\[ I_{12,2} = D_4 = G_1 + K_2 = G_{12} \oplus K_2 : [4, 2, 1^4, -1^4, -2^3, -4], \] also NEPS of \( C_3, K_2 \) and \( K_2 \) with basis \( \{(1, 1, 0), (0, 1, 1), (0, 0, 1)\} \), cf. [18].

\[ I_{12,3} : [4, 2, 1^4, 0, -1^2, -2^3, -3^2], \] cf. graph \( N^8 \) in Table 9.1 in \([4]\).

\[ I_{12,4} = I_{6,1} \times K_2 = D_3 = L(G_8) \times K_2 = C_3 \times C_4 : [4, 2^2, 0^6, -2^2, -4]. \]

\[ I_{12,5} = G_{12} + K_2 = C_3 + C_4 : [4, 2^2, 1^2, 0, -1^4, -3^2]. \]

\[ I_{12,6} = E_4 : [4, 2^1, 0^3, -2^5], \] cf. graph \( N^8 \) in Table 9.1 in \([4]\).

\[ I_{12,7} = E_5 = L(G_2) : [4, 2^4, 0^3, -2^5]. \]

\[ I_{12,8} : [4, 3, 1^3, 0, -1^3, -2^2, -3], \] cf. graph \( N^8 \) in Fig. 21 of \([1]\).

\[ I_{14,1} = I_{7,1} \times K_2 : [4, 3, 1^3, 0^4, -1^3, -3^2, -4]. \]

\[ I_{15,1} : [4, 2^4, 1, 0^2, -1^2, -2^4, -3], \] cf. graph \( X \) from \([17]\).

\[ I_{15,2} = E_8 = L(G_{10}) : [4, 2^5, -1^4, -2^2]. \]

\[ I_{15,3} = L(G_6) : [4, 3, 2^2, 1^2, 0^6, -1, -2^6]. \]

\[ I_{15,4} = L(G_{11}) : [4, 3, 2^3, 0^6, -1^3, -2^2]. \]

\[ I_{16,1} = I_{8,2} \times K_2 = D_5 = G_2 + K_2 = C_4 + C_4 : [4, 2^4, 0^6, -2^4, -4]. \]

\[ I_{16,2} = D_6 : [4, 2^4, 0^6, -2^4, -4]. \]

\[ I_{18,1} = I_{9,2} \times K_2 = D_{10} = L(G_1) \times K_2 = L(G_{12}) \times K_2 = (C_3 + C_3) \times K_2 : [4, 2^4, 1^4, -1^4, -2^4, -4]. \]

\[ I_{18,2} = D_{11} : [4, 2^5, 1^4, -1^4, -2^4, -4]. \]

\[ I_{18,3} = D_{12} : [4, 2^4, -1^4, -2^4, -4]. \]

\[ I_{18,4} = C_3 + C_6 : [4, 3^2, 1^4, 0^6, -2^4, -3^2]. \]

\[ I_{18,5} = L(G_7) : [4, 3^2, 2, 1^4, 0, -1^2, -2^7]. \]

\[ I_{18,6} = L(G_{13}) : [4, 3^3, 1^2, 0^3, -1^3, -2^6]. \]

\[ I_{19,1} = D_{13} = G_{10} \oplus K_2 : [4, 2^6, 1^4, -1^4, -2^5, -4]. \]

\[ I_{19,2} = D_{14} : [4, 2^6, 1^4, -1^4, -2^5, -4]. \]

\[ I_{19,3} = D_{15} : [4, 2^6, 1^4, -1^4, -2^5, -4]. \]

\[ I_{19,4} = G_{10} + K_2 : [4, 2^6, 0^3, -1^4, -3^2]. \]

\[ I_{19,5} = G_6 + K_2 : [4, 3, 2^3, 1^3, 0^4, -1^3, -2^3, -3, -4]. \]
\[ I_{28.6} = G_{11} \oplus K_2 : [4, 3, 2^3, 1^3, 0^4, -1^3, -2^3, -3, -4]. \]
\[ I_{28.7} = G_{11} + K_2 : [4, 3, 2^4, 1^5, 0^5, -1^3, -2^3, -3^3]. \]
\[ I_{28.8} = L_2(K_3) : [4, 3^3, 0^5, -1^4, -2^3]. \] It is proved in [7] that \( L_2(G) \) (\( G \) connected with at least two vertices) is integral if and only if \( G \) is a complete graph.

\[ I_{30.1} = D_7 : [4, 2^8, 0^6, -2^8, -4]. \]
\[ I_{30.2} = D_8 = L(G_7) \times K_2 : [4, 3^3, 0^6, -2^8, -4]. \]
\[ I_{30.3} = G_7 + K_2 = C_4 + C_6 : [4, 3^3, 2^2, 1^6, 0^7, -1^6, -2^2, -3^2, -4]. \]
\[ I_{30.4} = G_{13} \oplus K_2 : [4, 3^3, 1^5, 0^6, -1^5, -3^3, -4]. \]
\[ I_{30.5} = G_{13} + K_2 : [4, 3^3, 2^1, 1^5, 0^3, -1^5, -2^3, -3^3]. \]
\[ I_{31.1} = D_{16} = L(G_{18}) \times K_2 : [4, 2^{10}, 1^4, -1^4, -2^{10}, -4]. \]
\[ I_{31.2} = L(G_6) \times K_2 = I_{15.1} \times K_2 : [4, 3, 2^8, 1^3, 0^4, -1^3, -2^8, -3, -4]. \]
\[ I_{31.3} = L(G_{11}) \times K_2 : [4, 3, 2^8, 1^3, 0^4, -1^3, -2^8, -3, -4]. \]
\[ I_{31.4} = L(G_4) : [4, 3^3, 2^4, 0^6, -1^4, -2^4]. \]
\[ I_{31.5} = L(G_3) : [4, 3^3, 2^4, 0^6, -1^4, -2^4]. \]
\[ I_{32.1} = D_9 : [4, 2^{12}, 0^6, -2^{12}, -4]. \]
\[ I_{35.1} : [4, 2^{14}, -1^{14}, -3^6], \] the odd graph \( O_4 \) [14].
\[ I_{36.1} = L(G_7) \times K_2 : [4, 3^2, 2^8, 1^6, 0^2, -1^6, -2^8, -3^2, -4]. \]
\[ I_{36.2} = L(G_{13}) \times K_2 : [4, 3^5, 2^6, 1^5, 0^6, -1^5, -2^6, -3^3, -4]. \]
\[ I_{36.3} = C_6 + C_6 = (C_3 + C_3) \times K_2 : [4, 3^3, 2^4, 1^4, 0^{10}, -1^4, -2^4, -3^4, -4]. \]
\[ I_{36.4} = L(G_8) : [4, 3^6, 2^3, 1^4, 0^3, -1^6, -2^{13}]. \]
\[ I_{43.1} = G_4 + K_3 : [4, 3^4, 2^6, 1^4, 0^{10}, -1^4, -2^6, -3^4, -4]. \]

It is interesting to note that the graphs \( G_{10} + K_2, G_{11} + K_2 \) and \( L_2(K_3) \) all have mutually different spectra.

\[ I_{45.1} = L(G_3) : [4, 3^9, 1^{10}, -1^9, -2^4]. \]
\[ I_{48.1} = G_8 \times K_2 : [4, 3^6, 2^4, 1^{10}, 0^6, -1^{10}, -2^4, -3^6, -4]. \]
\[ I_{50.1} = L(G_4) \times K_2 : [4, 3^4, 2^6, 1^4, 0^{10}, -1^4, -2^6, -3^4, -4]. \]
\[ I_{50.2} = L(G_5) \times K_2 : [4, 3^4, 2^6, 1^4, 0^{10}, -1^4, -2^{14}, -3^4, -4]. \]
\[ I_{50.3} = G_3 + K_2 : [4, 3^5, 2, 1^{19}, -1^{19}, -2, -3^9, -4]. \]
\[ I_{70.1} = I_{35.1} \times K_2 : [4, 3^6, 2^{14}, 1^{14}, -1^{14}, -2^{14}, -3^6, -4]. \]
\[ I_{72.1} = L(G_8) \times K_2 : [4, 3^6, 2^6, 1^{10}, 0^6, -1^{10}, -2^{16}, -3^6, -4]. \]
\[ I_{78.1} = L(G_9) \times K_2 : [4, 3^9, 2^{16}, 1^{19}, -1^{19}, -2^{16}, -3^6, -4]. \]
5. CONCLUDING REMARKS

Possible spectra of connected 4-regular bipartite integral graphs, determined in Section 2, are quite numerous and we cannot expect that all 4-regular integral graphs will be determined in near future. Nonexistence results for many of these potential spectra will be obtained by considering appropriate graph properties but many others will remain to be considered.

Potential spectra from Subsection 1 (case $x = 0$), i.e. those from Tables 1 and 2, have been considered in [17] and all the corresponding graphs determined. They are graphs $D_1, \ldots, D_{18}$. Other known 4-regular integral graphs are presented in Section 4.

REFERENCES


---

University of Belgrade, Faculty of Electrical Engineering, P.O. Box 35-54, 11120 Belgrade, Yugoslavia
cvetkod@ubbg.etf.bg.ac.yu
esimics@ubbg.etf.bg.ac.yu

University of Niš, Department of Mathematics, Faculty of Philosophy, Cirila i Metodija 2, 18000 Niš, Yugoslavia
dragance@ni.ac.yu

(Received October 7, 1998)