Bipartite density of cubic graphs: the case of equality

Dragan Stevanović

Faculty of Science and Mathematics, Višegradska 33, 18000 Niš, Serbia and Montenegro

Abstract

Recently, Berman and Zhang (Discrete Math. 260 (2003), 27–35) obtained an upper bound for the bipartite density of cubic graphs in terms of the smallest eigenvalue of an adjacency matrix and attempted to characterize graphs for which the upper bound is attained. Here we do characterize graphs for which the upper bound is attained, and correct a few errors from Berman and Zhang’s paper.

Key words: Bipartite density of graph; Cubic graph; Eigenvalue; Line graph; Generalized line graph; Exceptional graph.

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Let $G$ be a simple graph with the set of vertices $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the set of edges $E(G)$. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$ and the $(0, 1)$-adjacency matrix of a graph $G$ is denoted by $A(G)$. The line graph $L(G)$ of a graph $G$ is defined by $V(L(G)) = E(G)$, where two vertices in $L(G)$ are adjacent if and only if they are adjacent as edges in $G$. For a non-negative integer $a$, the cocktail-party graph $CP(a)$ is defined as the unique $(2a - 2)$-regular graph on $2a$ vertices. Then the generalized line graph $L(G; a_1, a_2, \ldots, a_n)$, where $a_1, \ldots, a_n$ are non-negative integers, is obtained by taking the graph $L(G)$ and cocktail-party graphs $CP(a_i), \ i = 1, \ldots, n$, and adding extra edges: if a vertex $v_i$ is an end-vertex of an edge $e$ in $G$, then the vertex $e$ of $L(G)$ is joined to all vertices of $CP(a_i)$ in $L(G; a_1, a_2, \ldots, a_n)$. As a special case, we have that $L(G) \cong L(G; 0, 0, \ldots, 0)$.

Email address: dragance@pmf.ni.ac.yu (Dragan Stevanović).
URL: http://www.pmf.ni.ac.yu/dragance/ (Dragan Stevanović).

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The bipartite density $b(G)$ of $G$ is defined as

$$b(G) = \frac{|E(H)|}{|E(G)|},$$

where $H$ is any bipartite subgraph of $G$ with the maximum number of edges. Recently, Berman and Zhang published the following

**Theorem 1** [1, Theorem 3.5] Let $G$ be a connected cubic graph with $n$ vertices. If $\lambda_{\text{min}}$ is the smallest eigenvalue of the adjacency matrix $A(G)$ of $G$, then

$$b(G) \leq \frac{4}{7 + \lambda_{\text{min}}}. \quad (1)$$

Moreover, if $n \neq 20$, then equality holds if and only if $G$ is a bipartite graph, or the complete graph $K_4$, or the Petersen graph, or any of the four graphs in Fig. 1. \square

First, we should mention a small flaw in Fig. 1: namely, graph in its lower right part is not cubic. However, by deleting the thick edge it becomes cubic and then it does satisfy (1).

Further, Berman and Zhang did not characterize the case of equality when $n = 20$. Nowadays, that could have been done even by human-computer interaction: there are only 510, 489 cubic graphs on 20 vertices, which, for example, may be downloaded from [6], and then it would take a day or two to program and test this case (with $\approx 90\%$ of time for programming and only $\approx 10\%$ of time for testing).

Anyway, we will take more traditional approach and relate this case to some well developed topics of spectral graph theory.
Thus, suppose that the equality holds in (1) for a connected cubic graph $G$ with $n$ vertices. In [1], it is proved that any bipartite subgraph $H$ with the maximum number of edges must be either regular or semiregular. If $H$ is regular (cases 1 and 2 in the proof of Theorem 3.5 in [1]), then it is proved that $G$ is either a complete graph $K_4$ or a bipartite graph. If $H$ is semiregular (case 3 in the same proof), then it is proved that $b(G) = \frac{4}{5}$ and $\lambda_{\text{min}} = -2$, where $\lambda_{\text{min}}$ is the smallest eigenvalue of the adjacency matrix $A(G)$ of $G$. We discard the rest of the proof of case 3 of Theorem 3.5 in [1].

The classic result we use as a starting point is the following

**Theorem 2 ([3])** If $G$ is a graph with the smallest eigenvalue at least $-2$, then

(i) $G$ is a generalized line graph, or

(ii) $G$ is represented by one of root systems $E_6$, $E_7$ or $E_8$.

Suppose first that $G$ satisfies part (i) of Theorem 2, and that $G$ is isomorphic to a generalized line graph $L(G'; a_1, a_2, \ldots, a_m)$, for some graph $G'$ with the vertex set $\{u_1, u_2, \ldots, u_m\}$ and non-negative integers $a_1, a_2, \ldots, a_m$.

**Lemma 3** If $L(G'; a_1, a_2, \ldots, a_m)$ is cubic, then $a_1 = a_2 = \ldots = a_m = 0$.

**PROOF.** Suppose there exists $i$ such that $a_i \neq 0$. Then each vertex of $CP(a_i)$ is adjacent to each edge of $G'$, considered as a vertex of $L(G')$, having $u_i$ as an end-vertex, and, thus, degrees of vertices in $CP(a_i)$ are equal to $d_{G'}(u_i) + (2a_i - 2)$. Since $L(G'; a_1, a_2, \ldots, a_m)$ is cubic, it follows that $d_{G'}(u_i) + 2a_i = 5$ and, as a consequence, that $d_{G'}(u_i)$ is odd and, thus, positive.

Now, let $e$ be an edge of $G'$ having $u_i$ as an end-vertex, and let $u_j$ be another end-vertex of $e$. Edge $e$, considered as a vertex of $L(G')$, is adjacent to each vertex of $CP(a_i)$ and $CP(a_j)$, as well as to other edges of $G'$ incident with it. Thus, the degree of $e$ is equal to $(d_{G'}(u_i) + d_{G'}(u_j) - 2) + (2a_i + 2a_j)$, and since $d_{G'}(u_i) + 2a_i = 5$, $d_{G'}(u_j) \geq 1$ and $a_j \geq 0$, it follows that the degree of $e$ is at least 4, contradicting the assumption that $L(G'; a_1, a_2, \ldots, a_m)$ is cubic. \hfill $\square$

Thus, it follows that $G$ is a cubic line graph $L(G') \cong L(G'; 0, 0, \ldots, 0)$. However, in [1] it was shown that a cubic line graph with at least six vertices has bipartite density equal to $\frac{7}{9}$, contradicting the assumption that $b(G) = \frac{4}{5}$. Further, the only cubic line graph with at most five vertices is the complete graph $K_4$, which has bipartite density equal to $\frac{4}{6}$. Thus, $G$ must satisfy part (ii) of Theorem 2. A connected graph with the
The example from Figure 2 of [1].

The smallest eigenvalue at least \(-2\), which is not a generalized line graph, is called an exceptional graph. For a thorough survey of exceptional graphs, see [5]. The regular exceptional graphs have been found by a computer search in [2], and the cubic exceptional graphs have been found also theoretically in [4]. There are exactly five cubic exceptional graphs, each having 10 vertices, each with the smallest eigenvalue equal to \(-2\), and one of them being a Petersen graph. These graphs have been found also in [1] and those, other than a Petersen graph, are shown in Fig. 1 (with the thick edge deleted). It is easy to check that the bipartite density of each of these five graphs is equal to \(\frac{4}{5}\), and thus we have proved

**Lemma 4** Let \(G\) be a connected cubic graph with \(n\) vertices. If \(\lambda_{\text{min}}\) is the smallest eigenvalue of the adjacency matrix \(A(G)\) of \(G\), then the equality holds in (1) if and only if \(G\) is a bipartite graph, or the complete graph \(K_4\), or the Petersen graph, or any of the four graphs in Fig. 1 (with the thick edge deleted).

Together with Theorem 1 we have the following

**Theorem 5** Let \(G\) be a connected cubic graph with \(n\) vertices. If \(\lambda_{\text{min}}\) is the smallest eigenvalue of the adjacency matrix \(A(G)\) of \(G\), then

\[
b(G) \leq \frac{4}{7 + \lambda_{\text{min}}}. \tag{2}
\]

The equality holds if and only if \(G\) is a bipartite graph, or the complete graph \(K_4\), or the Petersen graph, or any of the four graphs in Fig. 1 (with the thick edge deleted).

Paper [1] contains one more error: in order to justify why they could not characterize the case of equality in (1) for graphs with 20 vertices, Berman and Zhang in Remark 3.6 of [1] gave an example of a graph (shown in Fig. 2) satisfying the equality, having bipartite density \(\frac{4}{5}\), with smallest eigenvalue \(-2\). However, while \(-2\) is an eigenvalue of this graph, the smallest eigenvalue is actually \(\approx -2.2559\), and thus this graph does not satisfy the equality in (1).
References


