A Note on Gutman’s Conjecture

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Let \( G = (V, E) \) be a simple graph, with the degree of a vertex \( i \in V \) denoted by \( d_i \). In [1] \( G \) is called \( SQR \) if \( (\sqrt{d_i})_{i \in V} \) is an eigenvector of (an adjacency matrix of) \( G \). Trivially, connected regular and semiregular graphs are \( SQR \) and Gutman conjectured in [1] that a connected \( SQR \) graph is either regular or semiregular.

A symmetric matrix \( A \) of order \( n \) all of whose entries are nonnegative, and all of whose row sums \( d_1, d_2, \ldots, d_n \) are positive, is called almost regular if there is a positive number \( r \) such that, if \( a_{ij} > 0 \), then \( d_id_j = r^2 \). Clearly, if \( G \) is connected and either regular or semiregular, then its adjacency matrix is almost regular.

Actually, Gutman’s conjecture may be generalized to say that if an irreducible nonnegative symmetric matrix \( A \) with positive row sums \( d_1, d_2, \ldots, d_n \) has an eigenvector \( (\sqrt{d_i})_{i=1}^n \) then \( A \) is almost regular.

The following theorem is proved in [2].

**Theorem 1 ([2])** Let \( A \) be a nonnegative symmetric matrix with positive row sums \( d_1, \ldots, d_n \) and set \( m = \sum_{i=1}^n d_i \). Then

\[
\rho(A) \geq \frac{\sum_{i,j} a_{ij} \sqrt{d_id_j}}{m} \geq \sqrt{\frac{m}{\sum_{i,j} \frac{a_{ij}}{d_id_j}}},
\]

where \( \rho(A) \) denotes the spectral radius of \( A \), and \( A \) is almost regular if and only if any two of these functions in \( A \) in (1) are equal, in which case all three are equal.

Proof of this theorem also contains proof of generalized Gutman’s conjecture. For the sake of completeness, we reproduce here just the corresponding part of the proof.

First, notice that \( (\sqrt{d_i})_{i=1}^n \) is a positive eigenvector of a nonnegative matrix \( A \), and thus, its corresponding eigenvalue is the spectral radius \( \rho(A) \). Now, if all the \( d_i \) are equal, we are done. Otherwise set \( \delta = \min\{d_1, \ldots, d_n\} \) and \( \Delta = \max\{d_1, \ldots, d_n\} \). Choose \( u \) and \( v \) such that \( d_u = \delta \) and \( d_v = \Delta \). Now assume that there exists \( w \) with \( a_{uw} > 0 \) and \( d_w < \Delta \). Then we have

\[
\rho(A) = \sum_{j=1}^n a_{uj} \sqrt{\frac{d_j}{\delta}} < \sum_{j=1}^n a_{uj} \sqrt{\frac{\Delta}{\delta}} = \sqrt{\delta \Delta}.
\]

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On the other hand,

$$\rho(A) = \sum_{j=1}^{n} a_{vj} \sqrt{d_j} \geq \sum_{j=1}^{n} a_{vj} \sqrt{\frac{\delta}{\Delta}} = \sqrt{\delta \Delta},$$

which is a contradiction.

Assuming existence of $w$ with $a_{vw} > 0$ and $d_w > \delta$ leads to an analogous contradiction. We conclude that whenever $a_{ij} > 0$ then $d_i = \delta$ and $d_j = \Delta$ or vice versa, and that $A$ is almost regular matrix with $\rho(A) = \sqrt{\delta \Delta}$.

References
