On Distance Integral Graphs

Milan Pokorný, Pavel Híc, Dragan Stevanović, Marko Milošević

Abstract

The distance eigenvalues of a connected graph $G$ are the eigenvalues of its distance matrix $D$, and they form the distance spectrum of $G$. A graph is called distance integral if its distance spectrum consists entirely of integers. We show that no nontrivial tree can be distance integral. We characterize distance integral graphs in the classes of graphs similar to complete split graphs, which, together with relations between graph operations and distance spectra, allows us to exhibit many infinite families of distance integral graphs.

Keywords: Distance Spectrum; Distance Integral Graph; Tree; Complete Split Graph.

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1. Introduction

A simple graph $G = (V, E)$ consists of the vertex set $V$ and the edge set $E$, where each edge in $E$ connects two distinct vertices in $V$. A walk between $u$ and $v$ in $G$ of length $k$ is a sequence of vertices and edges $W : u = w_0, e_1, w_1, e_2, ..., w_k = v$ where $w_0, w_1, ..., w_k ∈ W$ and edge $e_i$ connects vertices $w_{i-1}$ and $w_i$ for $i = 1, 2, ..., k$. The distance $d_G(u, v)$ between $u$ and $v$ in $G$ is the length of the shortest walk between $u$ and $v$, provided such a walk exists.

For a graph $G$, let $A(G)$ be an adjacency matrix of $G$. The characteristic polynomial is defined as $P(G; x) = |xI - A|$. A graph is called integral, if all the zeros of its characteristic polynomial are integers. The research for integral graphs started already in 1974 by Harary and Schwenk (see [8]) and has continued to this day so far, together with the research of distance integral graphs.
graphs, Laplacian integral graphs, signless Laplacian integral graphs and S-integral graphs. A survey of results on integral graphs up to 2001 is given by Balińska et al. (see [1]).

Let $n$ be a number of vertices of $G$. The distance matrix $D(G)$ is the $n \times n$ matrix, indexed by $V$, such that $(D(G))_{u,v} = d_G(u,v)$. The characteristic polynomial $P(G;x) = |xI - D(G)|$ is the distance characteristic polynomial of $G$. Since $D(G)$ is a real symmetric matrix, the distance characteristic polynomial has real zeros $\rho_1 \geq \rho_2 \geq ... \geq \rho_n$, which form the distance spectrum of $G$.

A graph $G$ is distance integral (briefly, D-integral) if its distance spectrum consists entirely of integers. Although there is a huge amount of papers that study distance spectrum of graphs and their applications to distance energy of graphs (see [2,6,9-15,17,18,20]), the D-integral graphs are studied only in [9, 15], in the case of some special, highly symmetric, graphs. For a survey on spectral properties of distance matrices of graphs see [16].

In the paper we study D-integral graphs. After giving an overview of preliminaries in Section 2, we give in Section 3 the list of connected D-integral graphs up to 10 vertices, their distance characteristic polynomials and some additional data. In Section 4 we show that no nontrivial tree is distance integral. Distance integral graphs in the classes of complete split graphs, multiple complete split-like graphs, extended complete split-like graphs and multiple extended complete split-like graphs are characterized in Section 5. Note that integral graphs in these classes were characterized by Hansen et al [7] and signless Laplacian integrality of these classes was given by Freitas et al [4]. In Section 6 we provide new infinite families of D-integral graphs based on graph operations.

2. Graph Compositions and Preliminary Results

In this part we give the definitions of well known graph operations that we use to construct infinite families of D-integral graphs.

**Definition 1.** For graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, let us define the following graphs on the Cartesian product $V_1 \times V_2$ as their vertex sets:

- in the sum (Cartesian product) $G_1 + G_2$ two vertices $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent if $(u_1 = v_1$ and $(u_2, v_2) \in E_2)$ or $(u_2 = v_2$ and $(u_1, v_1) \in E_1)$;
- in the lexicographic product $G_1[G_2]$, two vertices $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent if $(u_1, v_1) \in E_1$ or $(u_1 = v_1$ and $(u_2, v_2) \in E_2)$;
- in the strong product $G_1 \bigotimes G_2$ two vertices $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent if $((u_1, v_1) \in E_1$ and $(u_2 = v_2$ or $(u_2, v_2) \in E_2)$);
- in the conjunction $G_1 \bigodot G_2$ two vertices $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent if $(u_1, v_1) \in E_1$ and $(u_2, v_2) \in E_2$.

**Definition 2.** The join $G_1 \nabla G_2$ of graphs $G_1$ and $G_2$ is a graph obtained from the union of $G_1$ and $G_2$ by adding an edge joining every vertex of $G_1$ to every vertex of $G_2$.

In [19] $H$-join of graphs $G_1, G_2, ..., G_k$, where $H$ is an arbitrary graph of order $k$, is defined as follows.
Definition 3. [19] Let $H$ be a graph with $V(H) = \{1, 2, \ldots, k\}$, and $G_i$ be disjoint graphs of order $n_i (i = 1, 2, \ldots, k)$. Then the graph $\vee_H \{G_1, G_2, \ldots, G_k\}$ is formed by taking the graphs $G_1, G_2, \ldots, G_k$ and joining every vertex of $G_i$ to every vertex of $G_j$ whenever $i$ is adjacent to $j$ in $H$.

Note that if $H = P_2$ then the $P_2$-join of graphs $G_1$ and $G_2$ is $\vee_{P_2} \{G_1, G_2\} = G_1 \vee G_2$.

Definition 4. [7] For $a, b, n \in \mathbb{N}$, we have the following classes of graphs:
- the complete split graph $CS_a^b = K_n \setminus K_b$;
- the multiple complete split-like graph $MCSS_{b,n}^a = K_n \setminus nK_b$;
- the extended complete split-like graph $ECS_a^b = K_a \setminus (K_b + K_2)$;
- the multiple extended complete split-like graph $MECS_{b,n}^a = K_a \setminus n(K_b + K_2)$.

Definition 5. For the complete bipartite graph $K_{p,q}$ the graph $K_{p,q} \equiv K_{q,r}$ on $p + 2q + r$ vertices is obtained by adding the edges $\{(v_i, w_i) | i = 1, 2, \ldots, q\}$ from two disjoint graphs $K_{p,q}$ with vertex classes $V_1 = \{u_i | i = 1, 2, \ldots, p\}$, $V_2 = \{v_i | i = 1, 2, \ldots, q\}$ and $K_{q,r}$ with vertex classes $U_1 = \{u_i | i = 1, 2, \ldots, q\}$, $U_2 = \{z_i | i = 1, 2, \ldots, r\}$ respectively.

Definition 6. [10] The Hamming graph $Ham(m, n)$, $m \geq 2, n \geq 2$ of diameter $m$ and characteristic $n$ have vertex set consisting of all $m$-tuples of elements taken from an $n$-element set, with two vertices adjacent if and only if they differ in exactly one coordinate. $Ham(m, n) = K_n + K_n + \ldots + K_n$, the Cartesian product of $K_n$, the complete graph on $n$ vertices, $m$ times.

The following theorems give D-spectra of graphs constructed by above mentioned definitions.

Theorem 1. [18] For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and the eigenvalues $\lambda_{i,1} = r_i \geq \ldots \geq \lambda_{i,n_i}$ of the adjacency matrix of $G_i$. The distance spectrum of $G_1 \vee G_2$ consists of the eigenvalues $-\lambda_{i,j} - 2$ for $i = 1, 2$ and $j = 2, 3, \ldots, n_i$, and two more simple eigenvalues

$$n_1 + n_2 - 2 - \frac{r_1 + r_2}{2} \pm \sqrt{(n_1 - n_2 - \frac{r_1 - r_2}{2})^2 + n_1 n_2}.$$

Theorem 2. [10] Let $G$ and $H$ be two distance regular graphs on $p$ and $n$ vertices with distance regularity $k$ and $t$ respectively. Let $Spec_D(G) = \{k, \mu_2, \ldots, \mu_p\}$ and $Spec_D(H) = \{t, \eta_2, \ldots, \eta_n\}$. Then $Spec_D(G + H) = \{nk + pt, n\mu_i, p\eta_j, 0\}$, where $i = 2, \ldots, p$, $j = 2, \ldots, n$, and 0 is with multiplicity $(p-1)(n-1)$.

From Theorem 2 we have the following.

Corollary 3. a) Let $G$ and $H$ be two distance integral and distance regular graphs on $p$ and $n$ vertices with distance regularity $k$ and $t$ respectively. Let $Spec_D(G) = \{k, \mu_2, \ldots, \mu_p\}$ and $Spec_D(H) = \{t, \eta_2, \ldots, \eta_n\}$. Then $G+H$ is distance integral and its spectrum is $Spec_D(G + H) = \{nk + pt, n\mu_i, p\eta_j, 0\}$ for $i = 2, \ldots, p$, $j = 2, \ldots, n$ and 0 is with multiplicity $(p-1)(n-1)$. 

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b) Let $H$ be a distance integral with the spectrum $k = \rho_1 \geq \rho_2 \geq \ldots \geq \rho_p$ which is distance $k$-regular. Then $G = H + K_n$ is distance integral and its spectrum is $nk + p(n-1), n \cdot \rho_i$ for $i = 2, 3, \ldots, p$, $-p$ with multiplicity $n-1$ and $0$ with multiplicity $(p-1)(n-1)$.

c) Let $G = K_m + K_n$. Then $G$ is distance integral and its spectrum is $\text{Spec}_D(K_m + K_n) = \{(m-1)n + (n-1)m, -n, -m, 0\}$, $-n$ with multiplicity $m-1$, $-m$ with multiplicity $n-1$ and $0$ with multiplicity $(n-1)(m-1)$.

Theorem 4. [10] Let $G$ be a graph with $D$-matrix $D_G$ and $H$ be an $r$-regular graph with an adjacency matrix $A$. Let the distance spectrum of $G$ be $\text{Spec}_D(G) = \{\rho_1, \rho_2, \ldots, \rho_p\}$ and the adjacency spectrum of $H$ be $\{r, \lambda_2, \ldots, \lambda_n\}$. Then the distance spectrum of $G[H]$ consists of $n \cdot \rho_i + 2n - r - 2$ for $i = 1, 2, \ldots, p$ and $-\lambda_j - 2$ with multiplicity $p$ for $j = 2, \ldots, n-1$.

From Theorem 4 we have:

Corollary 5. Let $H$ be a distance integral with the spectrum $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_p$. Then $G = H[K_n]$ is distance integral and its spectrum is $n \cdot \rho_i + n - 1$ for $i = 1, 2, 3, \ldots, p$ and $-1$ with multiplicity $p(n-1)$.

Theorem 6. [10] Let $\text{Ham}(m, n)$ be the Hamming graph of characteristic $n$. Then $\text{Ham}(m, n)$ is $D$-integral for every positive integers $m$ and $n$ and its $D$-spectrum is $mn^{m-1}(n-1)$, $0$, and $-n^{m-1}$ with multiplicities 1, $n^{m} - m(n-1)$ and $m(n-1)$ respectively.

If we use $n = 2$ in theorem 6 we get the following corollary.

Corollary 7. Let $Q_m$ be a hypercube. Then $Q_m$ is distance integral for every $m \in N$ and its distance spectrum is $m \cdot 2^{m-1}$, $0$ with multiplicity $2^m - m - 1$, $-2^{m-1}$ with multiplicity $m$.

Theorem 8. [11] Let $G$ be an $r$-regular graph of diameter two with adjacency spectrum $r = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then the distance spectrum of the strong product $K_2 \bigotimes G$ consists of $5n - 2r - 4, 2r - n, -2(\lambda_i + 2)$ and $2\lambda_i$ for $i = 2, \ldots, n$.

From Theorem 8 we have:

Corollary 9. Let $H$ be an $r$-regular graph of diameter two with integral adjacency spectrum $r = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then $G = K_2 \bigotimes H$ is distance integral and its distance spectrum is $5n - 2r - 4, 2r - n, -2(\lambda_i + 2)$ and $2\lambda_i$ for $i = 2, 3, \ldots, n$.

3. Distance Integral Graphs on at Most 10 Vertices

We have found all connected $D$-integral graphs with up to ten vertices by a computer search. The numbers $c(n)$ of such graphs with $n$ vertices are given in Table 1. The graphs are listed in Table 2, together with their diameter and
the distance characteristic polynomial. From the description of graphs in this table, it turns out that most of them can be obtained by starting with complete graphs, empty graphs or cycles by applying (possibly several times) the join, the strong product, the conjunction or the sum of graphs. We were not able to find such representation for six graphs only: \( G_1, G_2 \) and its complement \( \overline{G_2} \), \( G_3 \), the Petersen graph and its complement. From this table one can also see that majority of \( D \)-integral graphs with up to ten vertices have diameter two, with only six graphs in this set with diameter three.

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_n )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>5</td>
<td>4</td>
<td>7</td>
<td>9</td>
<td>19</td>
</tr>
</tbody>
</table>

The graphs \( G_1, G_2 \) and \( \overline{G_2} \) from Table 2 are 4-regular graphs of order \( n = 9 \). The graph \( G_1 \) is formed from two closed cycles \( 1-2-3-4-5-6-7-8-9-1 \) and \( 1-6-4-8-2-9-3-7-5-1 \). Similarly, the graph \( G_2 \) is formed from two closed cycles \( 1-2-3-4-5-6-7-8-9-1 \) and \( 1-3-7-4-9-5-2-8-6-1 \). The graph \( G_3 \) from Table 2 is 6-regular graph of order \( n = 10 \). Its complement is formed from the closed cycle \( 1-2-3-4-5-6-7-8-9-10-1 \) and a perfect matching \((1,6), (2,4), (3,7), (5,9), (8,10)\).

4. There are no nontrivial distance integral trees

The computer search for \( D \)-integral connected graphs from the previous section revealed that there are only two trees among such graphs: \( K_1 \) and \( K_2 \). It was known already that trees have exactly one positive distance eigenvalue, with the remaining distance eigenvalues being negative [14]. The observed nonexistence of larger \( D \)-integral trees stems from the fact that every nontrivial tree has a distance eigenvalue in the interval \((-1,0)\).

**Theorem 10.** Every tree \( T \) with at least three vertices has a distance eigenvalue in the interval \((-1,0)\).

**Proof.** Let the edges of \( T \) be given an arbitrary orientation, and suppose that \( Q = (q_{ue}) \) is the vertex-edge incidence matrix of \( T \) such that \( q_{ue} = 1 \) if vertex \( u \) is the head of edge \( e \), \( q_{ue} = -1 \) if vertex \( u \) is the tail of \( e \), and \( q_{ue} = 0 \) otherwise. Then \( Q^TQ = 2I + A(T^*) \), where \( A(T^*) \) is the \((0,1)\)-adjacency matrix of the line graph of \( T \). The matrix \( K(T) = Q^TQ \) is closely related to the Laplacian matrix \( L(T) = QQ^T \), as \( K(T) \) and \( L(T) \) have identical nonzero parts of their spectra.

Merris [14] has proved that the eigenvalues of \(-2K^{-1}(T)\) interlace the eigenvalues of \( D(T) \). In particular, if \( d_2 \) is the second largest eigenvalue of \( D(T) \) and
Table 2: Connected D-integral graphs up to 10 vertices.

<table>
<thead>
<tr>
<th>No.</th>
<th>n</th>
<th>Diam.</th>
<th>Graph</th>
<th>Distance Characteristic Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>$K_2 = K_{1,1}$</td>
<td>$(x + 1)(x - 1)$</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td>$K_3 = K_{1,1,1}$</td>
<td>$(x + 1)^2(x - 2)$</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>2</td>
<td>$C_4 = K_{2,2} = K_2 + K_2 = K_2 \boxtimes K_2$</td>
<td>$(x + 2)^2(x - 4)$</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>1</td>
<td>$K_4$</td>
<td>$(x + 1)^3(x - 3)$</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>1</td>
<td>$K_5$</td>
<td>$(x + 1)^5(x - 4)$</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>3</td>
<td>$C_6 = K_3 \bullet K_2$</td>
<td>$(x + 4)^3(x + 1)x^2(x - 9)$</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>2</td>
<td>$K_{3,2} = K_3 \boxtimes K_2$</td>
<td>$(x + 2)^2(x - 7)$</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>2</td>
<td>$K_3 + K_2$</td>
<td>$(x + 3)(x + 2)x^2(x - 7)$</td>
</tr>
<tr>
<td>9</td>
<td>6</td>
<td>2</td>
<td>$K_{2,2,2} = K_3 \boxtimes K_2$</td>
<td>$(x + 2)^2x^2(x - 6)$</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>1</td>
<td>$K_6$</td>
<td>$(x + 1)^6(x - 5)$</td>
</tr>
<tr>
<td>11</td>
<td>7</td>
<td>2</td>
<td>$CS^1_7 = K_7 \setminus K_1 = K_{4,1,1,1}$</td>
<td>$(x + 2)^4(x + 1)x(x - 8)$</td>
</tr>
<tr>
<td>12</td>
<td>7</td>
<td>2</td>
<td>$CS^2_7 = K_7 \setminus K_2 = K_{4,2,1,1}$</td>
<td>$(x + 3)(x + 2)^2(x + 1)^2(x - 1)(x - 8)$</td>
</tr>
<tr>
<td>13</td>
<td>7</td>
<td>2</td>
<td>$CS^3_7 = K_7 \setminus K_3 = K_{4,3,1,1}$</td>
<td>$(x + 2)^2(x - 1)^2(x - 7)$</td>
</tr>
<tr>
<td>14</td>
<td>7</td>
<td>1</td>
<td>$K_7$</td>
<td>$(x + 1)^7(x - 6)$</td>
</tr>
<tr>
<td>15</td>
<td>8</td>
<td>3</td>
<td>$\nabla_{P_3}(K_2, K_2, K_2, K_2)$</td>
<td>$(x + 5)(x + 2)^2(x - 12)$</td>
</tr>
<tr>
<td>16</td>
<td>8</td>
<td>3</td>
<td>$C_4 + K_2 = K_4 \bullet K_2$</td>
<td>$(x + 2)^4x^3(x - 12)$</td>
</tr>
<tr>
<td>17</td>
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<td>$K_{4,4} = K_2 \boxtimes K_4 = C_4 \boxtimes K_2$</td>
<td>$(x + 2)^6(x - 10)$</td>
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<tr>
<td>18</td>
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<td>$K_4 + K_2$</td>
<td>$(x + 4)(x + 2)^3x^3(x - 10)$</td>
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<td>19</td>
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<td>$2K_2 \setminus K_2$</td>
<td>$(x + 3)^2(x - 1)^2(x - 9)$</td>
</tr>
<tr>
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<td>$K_{2,2,2,2} = K_4 \boxtimes K_2$</td>
<td>$(x + 2)^4x^2(x - 8)$</td>
</tr>
<tr>
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<td>$K_8$</td>
<td>$(x + 1)^8(x - 7)$</td>
</tr>
<tr>
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<td>$K_3 \setminus K_2$</td>
<td>$(x + 3)^2(x + 1)^2(x - 11)$</td>
</tr>
<tr>
<td>23</td>
<td>9</td>
<td>2</td>
<td>$G_1$</td>
<td>$(x + 4)(x + 3)(x + 2)^2(x - 1)^2x^3(x - 12)$</td>
</tr>
<tr>
<td>24</td>
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<td>25</td>
<td>9</td>
<td>2</td>
<td>$K_3 + K_3$</td>
<td>$(x + 3)^4x^2(x - 12)$</td>
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<td>26</td>
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<tr>
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<td>2</td>
<td>$K_{3,3,3} = K_3 \setminus K_2$</td>
<td>$(x + 2)^4x^3(x - 10)$</td>
</tr>
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<td>28</td>
<td>9</td>
<td>2</td>
<td>$ECS^3_3 = K_3 \setminus (K_3 \setminus K_2)$</td>
<td>$(x + 3)(x + 2)^4x^2(x - 10)$</td>
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<td>$(x + 2)^5x(x - 13)$</td>
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<td>$MCS^2_7, 2 = K_6 \setminus K_2$</td>
<td>$(x + 3)(x + 2)^4(x - 13)$</td>
</tr>
<tr>
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<td>$K_{2,3} = K_{3,2,2}$</td>
<td>$(x + 6)(x + 4)^2(x + 2)^2(x + 1)^2x^2(x - 17)$</td>
</tr>
<tr>
<td>34</td>
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<td>3</td>
<td>$K_{5,5} = M = K_5 \bullet K_2$</td>
<td>$(x + 4)^2x^4(x - 15)$</td>
</tr>
<tr>
<td>35</td>
<td>10</td>
<td>2</td>
<td>$MCS^3_7, 2 = K_7 \setminus (K_3 \setminus K_2)$</td>
<td>$(x + 4)(x + 2)^4(x + 1)^2x(x - 12)$</td>
</tr>
<tr>
<td>36</td>
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<td>$K_9 \setminus K_9$</td>
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<td>2</td>
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<td>38</td>
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<td>$\nabla_{P_3}(K_1, C_4, C_4, C_4)$</td>
<td>$(x + 4)(x + 2)^6(x + 1)^2x^2(x - 12)$</td>
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<td>$(x + 3)^3(x + 1)^2x^2(x - 11)$</td>
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<tr>
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<td>2</td>
<td>$K_{3,4} \setminus K_2$</td>
<td>$(x + 3)(x + 2)^4(x + 1)^4x(x - 11)$</td>
</tr>
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<td>10</td>
<td>2</td>
<td>$K_4 \setminus K_2$</td>
<td>$(x + 4)(x + 1)^4x(x - 11)$</td>
</tr>
<tr>
<td>42</td>
<td>10</td>
<td>2</td>
<td>$K_{2,2,2,2,2} = K_5 \setminus K_2$</td>
<td>$(x + 2)^3x^3(x - 10)$</td>
</tr>
<tr>
<td>43</td>
<td>10</td>
<td>2</td>
<td>$(K_3 + K_2) \setminus K_2$</td>
<td>$(x + 3)^3(x + 2)^4x^3(x - 11)$</td>
</tr>
<tr>
<td>44</td>
<td>10</td>
<td>2</td>
<td>$G_3$</td>
<td>$(x + 3)^4(x + 2)^2x^2(x - 12)$</td>
</tr>
<tr>
<td>45</td>
<td>10</td>
<td>2</td>
<td>$K_{2,2} = K_{2,2}$</td>
<td>$(x + 4)(x + 3)(x + 2)^2(x + 1)^2x^2(x - 12)$</td>
</tr>
<tr>
<td>46</td>
<td>10</td>
<td>2</td>
<td>complement to Petersen graph</td>
<td>$(x + 3)^2x(x - 12)$</td>
</tr>
<tr>
<td>47</td>
<td>10</td>
<td>2</td>
<td>$K_5 + K_2$</td>
<td>$(x + 5)(x + 2)^3x^2(x - 13)$</td>
</tr>
<tr>
<td>48</td>
<td>10</td>
<td>2</td>
<td>Petersen graph</td>
<td>$(x + 3)^2x^2(x - 15)$</td>
</tr>
<tr>
<td>49</td>
<td>10</td>
<td>1</td>
<td>$K_{10}$</td>
<td>$(x + 1)^9(x - 9)$</td>
</tr>
</tbody>
</table>
\(\lambda_1 \geq \lambda_2\) are two largest eigenvalues of \(K(T)\), then
\[
-\frac{2}{\lambda_1} \geq d_2 \geq -\frac{2}{\lambda_2}.
\] (1)

Grone et al [5, Corollary 4.3] have further proved that the number of Laplacian eigenvalues greater than two in a tree \(T\) with diameter \(d\) is at least \(\lfloor d/2 \rfloor\).

Therefore, theorem follows from (1) for any tree with diameter at least four, since \(\lambda_2 > 2\) in that case.

The remaining trees are divided in three special cases:

(a) If \(T\) has diameter two, then it is a star \(S_n = K_{1,n-1}\), whose distance spectrum by [18] consists of simple eigenvalues \(n - 2 \pm \sqrt{n^2 - 3n + 3}\) and
eigenvalue \(-2\) with multiplicity \(n - 2\). The distance eigenvalue \(n - 2 - \sqrt{n^2 - 3n + 3} \in (-1, 0)\) as

\[-1 = n - 2 - \sqrt{n^2 - 2n + 1} < n - 2 - \sqrt{n^2 - 3n + 3} < n - 2 - \sqrt{n^2 - 4n + 4} = 0.\]

(b) If \(T\) is a path \(P_4\) on four vertices (having diameter three), then it has a
distance eigenvalue \(\approx -0.58579\in (-1, 0)\).

(c) If \(T\) has diameter three, but it is not \(P_4\), then \(T\) contains the graph \(F\)
from Fig.1 as a subgraph.

![Figure 1: Subgraph of any tree of diameter three, different than \(P_4\).](image)

To conclude the proof, recall that for the Laplacian eigenvalues holds the
following form of the interlacing (see [5]): if \(e\) is an edge of a graph \(G\), then
\[
\lambda_i(G) \geq \lambda_i(G-e) \geq \lambda_{i+1}(G), \quad i = 1, \ldots, n-1.
\] (2)

Since \(F\) can be obtained from \(T\) by successively deleting edges from \(T\) (and noting that the possibly isolated vertices only influence the multiplicity of the smallest Laplacian eigenvalue \(0\)), it follows from (2) that
\[
\lambda_2(T) \geq \lambda_2(F) \approx 2.311111,
\]

and, as a consequence of (1), that \(d_2 \in (-1, 0)\). \(\square\)

5. Distance Integral Complete Split Graphs
In this part of the paper we give the characterization of D-integrality of complete split graphs.
Theorem 11.
1. For $a, b \in N$, the complete split graph $CS_b^a = \overline{K_a} \vee K_b$ has D-spectrum
   
   \[
   \left[\frac{2a+b-3}{2} \pm \frac{\sqrt{4a(a-1)+(b+1)^2}}{2}, -2a-1, -1^{b-1}\right].
   \]

2. For $a, b, n \in N$, the multiple complete split-like graph $MCS_{b,n}^a = \overline{K_a} \vee nK_b$ has D-spectrum
   
   \[
   \left[\frac{2a+(2n-1)b-3}{2} \pm \frac{\sqrt{4a(a-1)-b(n-1)^2}+[(2n-1)b+1]^2}}{2}, -2a-1, (-b-1)^{n-1}, -1^{n(b-1)}\right].
   \]

3. For $a, b \in N$, the extended complete split-like graph $ECS_b^a = \overline{K_a} \vee (K_b + K_2)$ has D-spectrum
   
   \[
   \left[\frac{2a+b-4}{2} \pm \frac{\sqrt{4a^2-4ab+9b^2}}{2}, -2a+2b-2, 0^{b-1}\right].
   \]

4. For $a, b, n \in N$, the multiple extended complete split-like graph $MECS_{b,n}^a = \overline{K_a} \vee n(K_b + K_2)$ has D-spectrum
   
   \[
   \left[\frac{2a+(4n-1)b-4}{2} \pm \frac{\sqrt{4a^2-4ab+(4n-1)^2b^2}}{2}, -2a+2b-2, 0^{n(b-1)}\right].
   \]

These graphs are distance integral if and only if the expressions under the square root are perfect squares.

PROOF. Using Theorem 1 we will prove the first part only. The proofs of the remaining parts are similar.

We use that $CS_b^a = \overline{K_a} \vee K_b$. The graph $\overline{K_a}$ is regular with degree 0 ($n_1 = a$, $r_1 = 0$), while the graph $K_b$ is regular with degree $b-1$ ($n_2 = b$, $r_2 = b-1$). The A-spectrum of $\overline{K_a}$ is $0^a$ and the A-spectrum of $K_b$ is $r_2 = b-1 \geq -1 \geq ... \geq -1$.

By Theorem 1 we have that the distance spectrum of $CS_b^a = \overline{K_a} \vee K_b$ consist of $-2^{a-1}$, $-1^{b-1}$ and two simple eigenvalues

\[
a + b - 2 - \frac{b-1}{2} \pm \sqrt{\left(a - b + \frac{b-1}{2}\right)^2 + ab}.
\]

Using routine simplification we get

\[
\frac{2a+b-3}{2} \pm \frac{\sqrt{4a(a-1)+(b+1)^2}}{2}.
\]

Since $2a+b-3$ and $4a(a-1)+(b+1)^2$ are integers of the same parity, (3) is integer if and only if $4a(a-1)+(b+1)^2$ is a perfect square. □

Corollary 12. The complete split graph $CS_b^a$ is integral if and only if there exists $p, q \in N$ with $(p,q) = 1$ and $c \in Z$ such that $a = p(\alpha + cq)$, $b = (\alpha + cq)(\beta + cp) - pq - 1$, where $\alpha, \beta \in Z$ are determined by the Euclidean algorithm such that $pa - q\beta = 1$. 

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Proof. The fact that $4a(a-1)+(b+1)^2$ is the perfect square means that there exists $k \in N$ such that $4a(a-1)+(b+1)^2 = (b+1+2k)^2$. After simplification we get $a(a-1) = k(b+1) + k^2$, from which we have

\[
\frac{a(a-1)}{k} - k - 1 = b
\]

and $k$ divides $a(a-1)$. Since the greatest common divisor $(a,a-1)$ of $a$ and $a-1$ is equal to 1, we have $k = (k,a(a-1)) = (k,a)(k,a-1)$.

Let $p = (k,a)$ and $q = (k,a-1)$. Thus $k = pq$, $(p,q) = 1$, $p|a$ and $q|a-1$. Let $\alpha, \beta \in Z$ be determined by the Euclidean algorithm such that $pa - q\beta = 1$ and let $a' = a - pa = a - 1 - q\beta$. From $p|a$ it follows that $p|a'$, while from $q|a-1$ it follows that $q|a'$. Since $(p,q) = 1$ we have that $pq|a'$ and $a = pq + cpq$ for some $c \in Z$, while $a-1 = q\beta + cpq$. Now from (4) it follows that $b = (\alpha + cq)(\beta + cp) - pq - 1$, $a = p(\alpha + cq)$.

**Corollary 13.** Let $a = k(t+1) + 1, b = t + 1, n = k + 1, k, t \in N$. Then the multiple complete split graph $MCS_{k,n}^a = K_n \nabla nK_n$ is $D$-integral.

Proof. It is sufficient to use the formula from Theorem 10, case 2. After simplification we get $4a[(a-1) - b(n-1)] + [(2n-1)b+1]^2 = [2k(t+1)+t+2]^2$.

**Corollary 14.**

a) The extended complete split-like graph $ECS_{k,n}^a = K_n \nabla (K_k + K_2)$ is $D$-integral if and only if

\[
a' = \frac{\alpha^2 + 2\alpha\beta - 8\beta^2}{2}, b' = 2\alpha\beta, a = a' \cdot q, b = b' \cdot q, q = \frac{k}{\text{gcd}(a',b')}
\]

for $\alpha, \beta, k \in N, \alpha > 2\beta$ with $\alpha$ even, or

\[
a' = \frac{-\alpha^2 + 2\alpha\beta + 8\beta^2}{2}, b' = 2\alpha\beta, a = a' \cdot q, b = b' \cdot q, q = \frac{k}{\text{gcd}(a',b')}
\]

for $\alpha, \beta, k \in N, \alpha < 4\beta$ with $\alpha$ even.

b) The multiple extended complete split-like graph $MMECS_{k,n}^a = K_n \nabla n(K_k + K_2)$ is $D$-integral if and only if

\[
a' = \frac{\alpha^2 + (16n - 8)\alpha\beta - (192n^2 - 64n)\beta^2}{2}, b' = 8\alpha\beta, a = a' \cdot q, b = b' \cdot q, q = \frac{k}{\text{gcd}(a',b')}
\]

for $\alpha, \beta, k \in N, \alpha > 8n\beta$ with $\alpha$ even, or

\[
a' = \frac{-\alpha^2 + (16n - 8)\alpha\beta + (192n^2 - 64n)\beta^2}{2}, b' = 8\alpha\beta, a = a' \cdot q, b = b' \cdot q, q = \frac{k}{\text{gcd}(a',b')}
\]

for $\alpha, \beta, k \in N, \alpha < (24n - 8)\beta$ with $\alpha$ even.
PROOF. We will prove only the case a), as the proof of the case b) is similar. It is easy to verify that $a, b$ satisfy the equation $4a^2 - 4ab + 9b^2 = c^2$. Now we will prove that there are no other solutions of $4a^2 - 4ab + 9b^2 = c^2$. Using the formula for solution of a quadratic equation we get $a_{1,2} = \frac{b \pm \sqrt{b^2 - 8c}}{2}$. Let us denote $c^2 - 8b^2 = p^2$. Then $a = \frac{b + p}{2}$.

As $8b^2 = c^2 - p^2$ we have $\frac{8b}{c - p} = \frac{c + p}{2} = \frac{2b}{p}$, from which we get $8\beta b = \alpha c - \alpha p$ and $\alpha b = \beta c + \beta p$.

Multiplying the first equation by $\alpha$ we get $8\alpha \beta b = \alpha^2 c - \alpha^2 p$. Multiplying the second equation by $8\beta$ we get $8\alpha \beta b = 8\beta^2 c + 8\beta^2 p$. So $\alpha^2 c - \alpha^2 p = 8\beta^2 c + 8\beta^2 p$, from which we get $\frac{c}{p} = \frac{\alpha^2 + 8\beta^2}{\alpha^2 - 8\beta^2}$.

Multiplying the first equation by $\beta$ we get $8\beta^2 b = \alpha \beta c - \alpha \beta p$. Multiplying the second equation by $\alpha$ we get $\alpha^2 b = \alpha \beta c + \alpha \beta p$. So $2\alpha \beta c = \alpha^2 b + 8\beta^2 b$, from which we get $\frac{c}{p} = \frac{2\alpha \beta}{\alpha^2 + 8\beta^2}$.

Now we have $p : c : b = (\alpha^2 - 8\beta^2) : (\alpha^2 + 8\beta^2) : 2\alpha \beta$. So $c = k(\alpha^2 + 8\beta^2)$, $b = 2k\alpha \beta$, $a = \frac{b + p}{2} = k\alpha^2 + 2\alpha \beta - 8\beta^2$ or $a = \frac{b - p}{2} = k - \alpha^2 + 2\alpha \beta + 8\beta^2$.

The condition $\alpha > 2\beta$ follows from $\alpha^2 + 2\alpha \beta - 8\beta^2 > 0$ and the condition $\alpha < 4\beta$ follows from $-\alpha^2 + 2\alpha \beta + 8\beta^2 > 0$. The condition $\alpha$ is even follows from $\alpha^2 + 2\alpha \beta - 8\beta^2$ has to be an integer, or $\alpha^2 + 2\alpha \beta + 8\beta^2$ has to be an integer. \(\Box\)

6. Other Infinite Families of D-integral Graphs

Now we will study the conditions for the graphs $K_a \nabla nK_b$, $nK_a \nabla nK_b$, $K_{a,a} \nabla nK_b$ and $(K_a + K_2) \nabla nK_b$ to be D-integral.

Theorem 15.

1. The graph $K_a \nabla nK_b$ has D-spectrum

   \[
   \left[ \frac{a + (2n - 1)b - 2}{2} \pm \frac{1}{2} \sqrt{a^2 + 2ab + (2n - 1)^2b^2}, -1^{n(b - 1)} + a^{-1}, (-b - 1)^{-1} \right].
   \]

2. The graph $nK_a \nabla nK_b$ has D-spectrum

   \[
   \left[ \frac{(2n - 1)(a + b) - 2}{2} \pm \frac{1}{2} \sqrt{2n - 1)^2(a - b)^2 + 4n^2ab}, \right.
   \]

   \[
   -1^{n(a + b - 2)}, (-a - 1)^{n - 1}, (-b - 1)^{n - 1}. \]

3. The graph $nK_a \nabla nK_a$ has D-spectrum

   \[
   \left[ 3na - a - 1, na - a - 1, (-a - 1)^{2n - 2}, -3^{2n(a - 1)} \right].
   \]

4. The graph $(K_a + K_2) \nabla nK_b$ has D-spectrum

   \[
   \left[ \frac{3a + (2n - 1)b - 3}{2} \pm \frac{1}{2} \sqrt{9a^2 + 2nab + 6ab - 6a + [2(n - 1)b + 1]^2}, \right.
   \]

   \[
   -a, -2^{a - 1}, 0^{a - 1}, (-b - 1)^{n - 1}, (-1)^{n(b - 1)} \right].
   \]
5. The graph $K_{a,a} \nabla nK_b$ has D-spectrum

\[
\left[ \frac{3a+(2n-1)b-3}{2} \pm \frac{1}{2} \sqrt{9a^2 - 4nab + 6ab - 6a + [(2n-1)b+1]^2 }, \right.
\]
\[
a - 2, -2^{2a-2}, (-b-1)^{n-1}, (-1)^{n(b-1)} .
\]

These graphs are distance integral if and only if the expressions under the square root are perfect squares.

PROOF. Using Theorem 1 we will prove only the case 1. The proofs of cases 2, 4 and 5 are similar and case 3 follows from case 2. The graph $K_a$ is regular on $a$ vertices with the degree $a - 1$ and its A-spectrum is $a - 1, -1^{a-1}$. The graph $nK_b$ is regular on $nb$ vertices with the degree $b - 1$ and its A-spectrum is $(b - 1)^n, -1^{n(b-1)}$.

By Theorem 1 we have that the distance spectrum of $K_{a,a} \nabla nK_b$ consist of $(-b-1)^{n-1}, -1^{n(b-1)+a-1}$ and two simple eigenvalues

\[
a + (2n-1)b - 2 \pm \frac{1}{2} \sqrt{a^2 + 2ab + (2n-1)^2b^2}.
\]  \hspace{1cm} (5)

Since $a + (2n-1)b - 2$ and $a^2 + 2ab + (2n-1)^2b^2$ are integers of the same parity, (5) is integer if and only if $a^2 + 2ab + (2n-1)^2b^2$ is a perfect square. □

Corollary 16.

a) The graph $K_{a,a} \nabla nK_b$ is D-integral if and only if

\[
a' = a^2 - 4\alpha\beta - (16n^2 - 16n)\beta^2, b' = 4\alpha\beta, a = a' \cdot q, b = b' \cdot q, q = \frac{k}{GCD(a', b')}\]

for $\alpha, \beta, k \in \mathbb{N}$ and $\alpha > 4n\beta$, or

\[
a' = -a^2 - 4\alpha\beta + (16n^2 - 16n)\beta^2, b' = 4\alpha\beta, a = a' \cdot q, b = b' \cdot q, q = \frac{k}{GCD(a', b')}\]

for $\alpha, \beta, k \in \mathbb{N}$ and $\alpha < (4n-4)\beta$.

b) The graph $nK_a \nabla nK_b$ is D-integral if and only if

\[
a' = \frac{a^2 + 4(2n^2 - 4n + 1)\alpha\beta - 16n^2(3n^2 - 4n + 1)\beta^2}{2n-1},
\]
\[
b' = (8n-4)\alpha\beta, a = a' \cdot q, b = b' \cdot q, q = \frac{k}{GCD(a', b')}\]

for $\alpha, \beta, k \in \mathbb{N}$ and $\alpha > 4n^2\beta$, or

\[
a' = -a^2 + 4(2n^2 - 4n + 1)\alpha\beta + 16n^2(3n^2 - 4n + 1)\beta^2, \]
\[
b' = (8n-4)\alpha\beta, a = a' \cdot q, b = b' \cdot q, q = \frac{k}{GCD(a', b')}\]

for $\alpha, \beta, k \in \mathbb{N}$ and $\alpha < (12n^2 - 16n + 4)\beta$. 

\hspace{1cm} 11
Proof. The proofs are similar to those of Corollary 13.

**Corollary 17.** Let \( a = kt, b = (2k - 1)t - 1, n = k + 1, k, t \in \mathbb{N} \). Then both \((K_a + K_2) \nabla nK_b\) and \(K_a, a \nabla nK_b\) are D-integral.

Proof. It is sufficient to use the formula from Theorem 14, cases 4 and 5. After simplification we get \(9a^2 - 4nab + 6a + [(2n - 1)b + 1]^2 = [4k^2t - k(t+2) + t]^2\). □

Now we give some new classes of D-integral graphs constructed by graph operations.

**Theorem 18.** Let \( G \) be a graph with distance spectrum \( \text{Spec}_D(G) = \{\rho_1, \rho_2, \ldots, \rho_p\} \). Then the distance spectrum of \( G \otimes K_n \) consists of \( n \cdot \rho_i + 2(n - 1) \) for \( i = 1, 2, \ldots, p \) and \(-2\) with multiplicity \((n - 1)p\).

Proof. Let \( D = D(G) \) be the distance matrix of \( G \). By definition of \( G \otimes K_n \) the vertex set \( V(G \otimes K_n) = \bigcup_{i=1}^n \{v_{i1}, v_{i2}, \ldots, v_{ip}\} \). The distance matrix of \( G \otimes K_n \) has the form

\[
\begin{pmatrix}
D & D + 2I_p & D + 2I_p & \cdots & D + 2I_p \\
D + 2I_p & D & D + 2I_p & \cdots & D + 2I_p \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
D + 2I_p & D + 2I_p & D + 2I_p & \cdots & D
\end{pmatrix}
\]

and the distance characteristic polynomial is

\[
P(G \otimes K_n; x) = \left| \begin{array}{cccccccc}
xI_p & -D - 2I_p & D + 2I_p & \cdots & D + 2I_p \\
-D - 2I_p & xI_p & -D - 2I_p & \cdots & D + 2I_p \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-D - 2I_p & -D - 2I_p & D + 2I_p & \cdots & xI_p - D
\end{array} \right|.
\]

Subtracting the last row from the previous ones we get

\[
P(G \otimes K_n; x) = \left| \begin{array}{cccccccc}
xI_p + 2I_p & 0 & 0 & \cdots & -xI_p - 2I_p \\
0 & xI_p + 2I_p & 0 & \cdots & -xI_p - 2I_p \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-D - 2I_p & -D - 2I_p & -D - 2I_p & \cdots & xI_p - D
\end{array} \right|.
\]

and adding the first \( n - 1 \) columns to the last one we get

\[
P(G \otimes K_n; x) = \left| \begin{array}{cccccccc}
xI_p + 2I_p & 0 & 0 & \cdots & 0 \\
0 & xI_p + 2I_p & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-D - 2I_p & -D - 2I_p & -D - 2I_p & \cdots & xI_p - 2(n-1)I_p - nD
\end{array} \right|.
\]
Since $P(G \boxtimes K_n; x) = det(xI_p + 2I_p)^{n-1}det(xI_p - 2(n-1)I_p - nD)$, the distance spectrum $spec_D(G \boxtimes K_n)$ consists of $n \cdot \rho_i + 2(n-1)$ for $i = 1, 2, ..., p$ and $-2$ with multiplicity $(n-1)p$.

Corollary 19. Let $H$ be a distance integral with the spectrum $\rho_1 \geq \rho_2 \geq ... \geq \rho_p$. Then $G = H \boxtimes K_n$ is distance integral and its spectrum is $n \cdot \rho_i + 2(n-1)$ for $i = 1, 2, 3, ..., p$ and $-2$ with multiplicity $(n-1)p$.

Lemma 20. [3] Let 

$$A = \begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$$

be $2 \times 2$ block symmetric matrix. Then the eigenvalues of $A$ are those of $A_0 + A_1$ together with those of $A_0 - A_1$.

Theorem 21. The graph $K_n \bullet K_2 = (K_{n,n} - M_n)$, where $M_n$ be a perfect matching on $2n$ vertices, is a distance integral and its distance spectrum is $(-4)^{n-1}, 0^{n-1}, n-4, 3n$.

Proof. Distance matrix of $K_n \bullet K_2$ has the form

$$A_d = \begin{pmatrix} 2A(K_n) & J_{n,n} + 2I_n \\ J_{n,n} + 2I_n & 2A(K_n) \end{pmatrix}.$$ 

By Lemma 20 the spectrum of $A_d$ consists of the spectrum of $3(J_{n,n})$ and the spectrum of $A(K_n) - 3I_n$. The distance polynomial of $3(J_{n,n})$ has the form $|xI_n - J_{n,n}| = x^{n-1}(x-n+1)$. Similarly the distance polynomial of $A(K_n) - 3I_n$ has the form $|xI_n - A(K_n) + 3I_n| = (x+4)^{n-1}(x-n+4)$.

Theorem 22. Let $G = K_{n-1,n} \equiv K_{n,n-1}$. Then $G$ is $D$-integral for every $n \in N$ and its distance spectrum is $\{-2n, -4^{n-1}, -2^{n-2}, -1, 0^{n-1}, 2n-4, 8n-7\}$.

Proof. The distance matrix of the graph is

$$D(K_{n-1,n} \equiv K_{n,n-1}) = \begin{pmatrix} 2K_{n-1,n-1} & J_{n-1,n} & 2J_{n-1,n} & 3J_{n-1,n-1} \\ J_{n,n-1} & 2K_{n,n} & J_{n,n-1} & 2K_{n,n-1} \\ 2J_{n,n-1} & I_0 + 3K_{n,n} & 2K_{n,n} & J_{n,n-1} \\ 3J_{n-1,n-1} & 2J_{n-1,n} & J_{n-1,n} & 2K_{n-1,n-1} \end{pmatrix}.$$ 

The distance characteristic polynomial of $G = K_{n-1,n} \equiv K_{n,n-1}$ has the form

$$P(G; x) = \begin{vmatrix} xI_{n-1} - 2K_{n-1,n-1} & -J_{n-1,n} & -2J_{n-1,n} & -3J_{n-1,n-1} \\ -J_{n,n-1} & xI_{n} - 2K_{n,n} & -I_{n} - 3K_{n,n} & -2J_{n,n-1} \\ -2J_{n,n-1} & -I_{n} - 3K_{n,n} & xI_{n} - 2K_{n,n} & -J_{n,n-1} \\ -3J_{n-1,n-1} & -2J_{n-1,n} & -J_{n-1,n} & xI_{n-1} - 2K_{n-1,n-1} \end{vmatrix}.$$ 

Using routine simplification we get

$$P(K_{n-1,n} \equiv K_{n,n-1}; x) = (x - 8n + 7)(x - 2n + 4)(x + 2)2^{n-4}(x + 4)^{n-1}(x + 1)(x + 2n)x^{n-1}.$$ 

□

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7. Conclusion

We have studied here some classes of distance integral graphs. We have presented a complete list of distance integral graphs on at most ten vertices, we show that there are no nontrivial distance integral trees, we characterize distance integrality of complete split graphs, multiple complete split-like graphs, extended complete split-like graphs and multiple extended complete split-like graphs and we give the new infinite families of distance integral graphs constructed using join, strong product and conjunction of graphs.

Among the classes of graphs obtained by the join of graphs, we have resolved many questions related to their distance integrality. However, two questions remain open:

**Question 1.** Characterize distance integral multiple complete split graphs $MCS_{a,b}^n$.

**Question 2.** Characterize distance integral graphs of the forms $(K_a + K_2)\nabla nK_b$ and $K_{a,a} \nabla nK_b$.

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References


