Two new and shorter proofs in graph theory

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Abstract
We give two new and shorter proofs of known results in graph theory.

1 Introduction
While studying a mathematical problem, one sometimes finds new relations
which make possible to find shorter or more obvious or more elegant proofs of
known results. Here we present two such proofs in graph theory. The results
in the first section are found by the first author, and the proofs in the second
section by the second author.

2 Some spectral bounds on metric graph invariants

Let
\[ A = \sum_{i=1}^{m} \mu_i P_i \]
be the spectral decomposition of the adjacency matrix of some connected graph,
say \( G \). Then the powers of \( A \) are
\[ A^k = \sum_{i=1}^{m} \mu_i^k P_i \quad (k = 0, 1, 2, \ldots). \]

Let \( N_k(s, t) \) be the number of walks of length \( k \) between the vertices \( s \) and \( t \).
Then
\[ N_k(s, t) = \sum_{i=1}^{m} \mu_i^k (P_i)_{st}. \]

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Therefrom, \(d(s,t) = \min\{k \mid N_k(s,t) \neq 0\}\). Let \(m_{st}\) be the number of non-zero members in the list: \((P_1)_{st}, \ldots, (P_m)_{st}\). Then the following lemma holds.

**Lemma 1** If \(s\) and \(t\) are two vertices of \(G\) then \(d(s,t) \leq m_{st} - 1\).

**Proof** For short, put \(d = d(s,t)\), and assume to the contrary, that \(d \geq m_{st}\). Then

\[
\sum_{i=1}^{m} \mu_i^k(P_{st}) = 0 \quad (k = 0,1,\ldots,d-1)
\]

while

\[
\sum_{i=1}^{m} \mu_i^1(P_{st}) \neq 0.
\]

If we regard (1) as a system of linear equations in \(m_{st}\) unknowns (in fact, in non-zero members from the list \((P_1)_{st}, \ldots, (P_m)_{st}\); recall \(m - m_{st}\) members are equal to zero), then, by our assumption, we get that \((P_i)_{st} = 0\) for all \(i = 1,\ldots,m\). But this contradicts (2). \(\Box\)

We will next try to find some bounds on \(m_{st}\). Observe first that \((P_i)_{st}\) is equal to 0 if (i) \(||P_i e_s|| = 0\), or (ii) \(||P_i e_t|| = 0\), or (iii) if these two vectors are orthogonal. Consider now the first two conditions, i.e. (i) and (ii). Recall also that \(||P_i e_j|| = \alpha_{ij}\) (here \(\alpha_{ij}\) is the \((i,j)\)-th entry of the angle matrix \(A = (\alpha_{ij})\)-index \(i\) corresponds to the \(i\)-th eigenspace, while \(j\) to the vertex \(j\) of the graph in question; note also that the angle matrix is usually ordered so that it represents a graph invariant, cf. [?] p. ?). Now, if \(\alpha_{is} = 0\) or \(\alpha_{it} = 0\), then \((P_i)_{st} = 0\). Based on this fact we define \(\hat{m}_{st} = \{i \mid \alpha_{is} \neq 0 \land \alpha_{it} \neq 0\}\). Then we arrive to the following result:

**Lemma 2** If \(s\) and \(t\) are two vertices of \(G\) then \(m_{st} \leq \hat{m}_{st}\).

**Remark** In view of (iii), in general, \(m_{st} \neq \hat{m}_{st}\).

We now switch to the bounds on vertex eccentricities. Recall, \(ecc(s) = \max\{d(s,t) \mid t \in V(G)\}\). By Lemma 1 we first get that \(ecc(s) \leq \max\{m_{st} \mid t \neq s\} - 1\); by Lemma 2 we next get that \(ecc(s) \leq \max\{\hat{m}_{st} \mid t \neq s\} - 1\). Let \(m_s = \max\{m_{st} \mid t \neq s\}\). So we arrive to the following result:

**Theorem 3** If \(s\) is any vertex of \(G\), then \(ecc(s) \leq m_s - 1\).

Let \(m'_s\) be the number of non-zero entries in the \(s\)-th column of the angle matrix of \(G\). Clearly, \(m_s \leq m'_s\) (since only condition (i) has been used); note also that \(m'_s = m_{ss}\). So we get the following result of C. Godsil (cf. [?]) as a corollary:

**Corollary 4** If \(s\) is any vertex of \(G\), then \(ecc(s) \leq m'_s - 1\).

**Remark** It is well known that \(diam(G) \leq m - 1\) for any connected graph (see [?], p. ?). Based on these observations we have \(diam(G) \leq \max\{m_s \mid s \in V(G)\} - 1\). In many situations this bound can be better than the previous one. Note also that this bound is completely extracted from the angle matrix of a graph (in contrast to bounds, see [?] p. ?, where the eigenvalues are used as well alpha ).
3 A new proof of the Runge-Hofmeister conjecture

Let $G$ be a simple undirected graph with $n$ vertices and $m$ edges; as usual, the vertex set and the edge set are denoted by $V$ and $E$, respectively. The spectral radius $r_\sigma$ of $G$ is the largest eigenvalue of the adjacency matrix $A$ of $G$. The degree of a vertex $i \in V$ is denoted by $d_i$.

The graph $G$ is called $(r_1, r_2)$-semiregular, if it is bipartite with bipartition $(V_1, V_2)$ such that all vertices in $V_i$ have the same degree $r_i$ for $i = 1, 2$. The graph $G$ is called almost regular, if there is a nonnegative real number $r$ such that every component of $G$ is either $r$-regular or $(r_1, r_2)$-semiregular with $r_1 r_2 = r^2$. Runge [4] (see also [1, p. 49]) showed that if $G$ is regular or semiregular graph, then

$$m = r_\sigma^2 \sum_{(i,j) \in E} \frac{1}{d_i d_j},$$

and conjectured that the condition (3) is sufficient for a graph to be regular or semiregular. Since Runge did not explicitly mentioned that a graph needs to be connected, Hofmeister [3] weakened this to conjecture that the condition (3) is sufficient for a graph to be almost regular. This conjecture was proved in [2], and here we give a new and shorter proof.

**Theorem 1** For a graph $G$ holds

$$r_\sigma^2 \geq \frac{m}{\sum_{(i,j) \in E} \frac{1}{d_i d_j}}.$$  

The case of equality holds if and only if $G$ is almost regular.

**Proof** It is known that

$$r_\sigma = \sup \{ x^T A x : x \in \mathbb{R}^n, \|x\| = 1 \}$$

$$= \sup \{ 2 \sum_{(i,j) \in E} x_i x_j : x \in \mathbb{R}^n, \|x\| = 1 \},$$

with the supremum attained if and only if $x$ is an eigenvector of $A$ corresponding to $r_\sigma$. Setting $x_i = \sqrt{\frac{d_i}{2m}}$ for $i \in V$ (giving $\|x\| = 1$, because $\sum_{i \in V} d_i = 2m$), from (5) we get

$$r_\sigma \geq 2 \sum_{(i,j) \in E} \sqrt{\frac{d_i}{2m}} \sqrt{\frac{d_j}{2m}} = \frac{1}{m} \sum_{(i,j) \in E} \sqrt{d_i d_j}.$$  

From the inequality between arithmetic and geometric means, we get

$$r_\sigma \geq \left( \prod_{(i,j) \in E} \sqrt{d_i d_j} \right)^{1/m}.$$
Squaring both sides of previous inequality, we get

$$r^2_\sigma \geq \left( \prod_{(i,j) \in E} d_id_j \right)^{1/m}.$$ 

Finally, from the inequality between geometric and harmonic means, we get

$$r^2_\sigma \geq \frac{m}{\sum_{(i,j) \in E} \frac{1}{d_id_j}}.$$  \hspace{1cm} (6)

The case of equality in (6) holds if and only if $x = (x_i)_{i \in V}$ is an eigenvector of $A$ corresponding to $r_\sigma$ and

$$d_id_j = r^2_\sigma \text{ for each } (i, j) \in E.$$  \hspace{1cm} (7)

Let $u$ be any vertex of $G$, and let $C$ be a component of $G$ containing $u$. From (7) we conclude that for all vertices $v$ of $C$ at odd distance from $u$ holds $d_v = r^2_\sigma d_u$, while for all vertices $v$ of $C$ at even distance from $u$ holds $d_v = d_u$. Thus, if $d_u = r_\sigma$ then $C$ is $r_\sigma$-regular graph, while if $d_u \neq r_\sigma$ then $C$ is $(d_u, \frac{r^2_\sigma}{d_u})$-semiregular graph with bipartition $(\{v \in C \mid d_v = d_u\}, \{v \in C \mid d_v = \frac{r^2_\sigma}{d_u}\})$.

References


