Semiharmonic bicyclic graphs*

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Abstract

Classification of harmonic and semiharmonic graphs according to their cyclomatic number became of interest recently. All finite harmonic graphs with up to four independent cycles, as well as all finite semiharmonic graphs with at most one cycle were determined. Here, we determine all finite semiharmonic bicyclic graphs. Besides that, we present several methods to construct semiharmonic graphs from existing ones, and we apply one of these constructions to show that the number of semiharmonic graphs with fixed cyclomatic number $k$ is infinite for every $k$.

MSC Classification: 05C75

Keywords: Walks in graphs, harmonic graphs, semiharmonic graphs, bicyclic graphs, constructions of new graphs

1 Some Standard Notations and Definitions

We consider locally finite simple graphs $G = (V, E)$ with vertex set $V = V_G$, edge set $E = E_G \subseteq (V^2)$, and adjacency matrix $A = A_G$ (indexed by the elements of $V$). The degree of a vertex $v \in V$ and the set of its neighbors are denoted by $d(v) = d_G(v)$ and $N(v) = N_G(v)$, respectively. We suppose that $G$ is connected and that $E_G$ is non-empty.

*Both authors would like to thank the DFG for support, while the second author was also supported by Grant 1227 of the Serbian Ministry of Science, Technology and Development.
The number of walks of length \( k \) of \( G \) starting at \( v \) is denoted by \( d_k(v) \). Clearly, one has

\[
d_0(v) = 1, \quad d_1(v) = d(v) \quad \text{and} \quad d_{k+1}(v) = \sum_{w \in N(v)} d_k(w).
\]

If \( G \) is finite, the number of all walks in \( G \) of length \( k \) is denoted by \( W_k = W_k(G) \). So, we have also

\[
W_0 = \#V, \quad W_1 = 2\#E, \quad \text{and} \quad W_k = \sum_{v \in V} d_k(v)
\]

for every \( k \in \mathbb{N}_0 \). In view of the well-known fact that, with \( j = j_V \) denoting the all-one vector defined on the set \( V \), the vector \((d_k(v))_{v \in V}\) coincides with \( A_k j \), we also have

\[
W_k = j^T A_k j, \quad (1)
\]

for all \( k \in \mathbb{N}_0 \).

A graph \( G \) is called regular if there exists a constant \( r \) such that \( d(v) = r \) holds for every \( v \in V \) in which case \( G \) is also called \( r \)-regular. Obviously, this is equivalent with the assertion that \( A_j = r_j \) holds. Further, a graph \( G \) is called \( a,b \)-semiregular if \( \{d(v), d(w)\} = \{a, b\} \) holds for all edges \( \{v, w\} \in E \). Clearly, this implies \( A^2 j = ab j \).

The harmonic graphs form another class of graphs: A graph \( G \) is called harmonic if there exists a constant \( \mu \) such that \( d_2(v) = \mu d(v) \) holds for every \( v \in V \) in which case \( G \) is also called \( \mu \)-harmonic. Clearly, a graph \( G \) is \( \mu \)-harmonic if and only if \( A^2 j = \mu A j \) holds.

Finally, a graph \( G \) is called semiharmonic if there exists a constant \( \mu \) such that \( d_3(v) = \mu d(v) \) holds for every \( v \in V \) in which case the graph \( G \) is also called \( \mu \)-semiharmonic. Clearly, the assertion that \( G \) is \( \mu \)-semiharmonic is equivalent with the assertion that \( A^3 j = \mu A j \) holds. Thus, every \( \mu \)-harmonic graph is \( \mu^2 \)-semiharmonic. Also, every \( a,b \)-semiregular graph is \( ab \)-semiharmonic. Moreover, every graph \( G \) for which some integers \( k, l \) with \( 0 \leq l < k \) and some constant \( \mu \) with \( d_k(v) = \mu d_l(v) \) for all \( v \in V \) exist, is semiharmonic — and even harmonic in case \( k - l \) is odd (cf. [6]). A semiharmonic graph that is not harmonic will henceforth be called strictly semiharmonic.

Harmonic graphs appeared for the first time in [10] where they were called dual-degree-regular graphs. Almost ten years later, they reappeared in [6, 7] under the present name and in connection with counting walks in a graph. All (finite or infinite) harmonic trees were constructed in [11]. All finite harmonic graphs with up to four independent cycles were characterized in [1] where it was also shown that, while the number of finite harmonic trees is infinite, the number of finite harmonic graphs with a fixed positive cyclomatic number is
finite. Semiharmonic graphs with cyclomatic number 0 or 1 were characterized in [4].

Here, we continue this work and determine all semiharmonic bicyclic graphs. It turns out that, while a semiharmonic unicyclic graph may have arbitrarily large cycle length, a semiharmonic bicyclic graph belongs to one of six classes with prescribed cycle structure. In addition, we present several methods to construct new semiharmonic graphs from existing ones that are used to show that, in contrast to harmonic graphs, the number of semiharmonic graphs with a fixed cyclomatic number is always infinite.

For our proofs, we need a corollary of the following lemma a proof of which can be found in [9]:

**Lemma 1** Given a locally finite semiharmonic graph $G$ and two vertices $v, v'$ that are both adjacent to a third vertex $u$, one has $d_2(v)d(v') = d_2(v')d(v)$.

**Corollary 1** Any finite, connected, and strictly semiharmonic graph $G$ is bipartite, and the average degree

$$ad(v) := \frac{d_2(v)}{d(v)}$$

of the neighbors of a vertex $v \in V$ is constant on each bipartite class of $G$.

Moreover, if $\{u_0, v_0\} \in E$, then we have

$$d_3(u_0) = \sum_{v \in N(u_0)} d_2(v) = \sum_{v \in N(u_0)} \frac{d_2(v_0)}{d(v_0)} d(v) = \frac{d_2(v_0)}{d(v_0)} d_2(u_0) = \frac{d_2(v_0)}{d(v_0)} \frac{d_2(u_0)}{d(u_0)} d(u_0),$$

so the ratio $d_3(u)/d_2(u)$ is also constant on both bipartite classes and coincides, for any given $u$ in one class, with the corresponding ratio $d_2(v)/d(v)$ for all $v \in V$ in the other class; in particular, $G$ is $(\frac{d_2(u_0)}{d(u_0)} \cdot \frac{d_2(v_0)}{d(v_0)})$-semiharmonic for all $\{u_0, v_0\} \in E$.

**2 Constructions of new semiharmonic graphs**

In this section, we present several methods for constructing semiharmonic graphs. The first two constructions exploit products and sums of graphs.

Given two graphs $G = (V, E)$ and $H = (U, F)$, the product $G \times H$ of $G$ and $H$ is the graph with vertex set $V \times U$ and edge set $E \otimes F := \{(v_1, u_1), (v_2, u_2)\} \subseteq V \times U \mid \{v_1, v_2\} \in E \text{ and } \{u_1, u_2\} \in F$, and the sum $G + H$ of $G$ and $H$ is the graph with vertex set $V \times U$ and edge set $E \oplus F := \{(v_1, u_1), (v_2, u_2)\} \subseteq V \times U \mid \{v_1, v_2\} \in E \text{ and } u_1 = u_2$, or $v_1 = v_2$ and $\{u_1, u_2\} \in F$. 
Further, given two square matrices $A = (a_{v_1,v_2})_{v_1,v_2 \in V}$ and $B = (b_{u_1,u_2})_{u_1,u_2 \in U}$ indexed by the sets $V$ and $U$, respectively, the tensor product $A \otimes B$ of the two matrices $A$ and $B$ is the square matrix $(a_{v_1,v_2} \cdot b_{u_1,u_2})_{(v_1,u_1),(v_2,u_2) \in V \times U}$ indexed by the set $V \times U$. Using this notation, we have $A_{G \times H} = A_G \otimes A_H$ and $A_{G \cup H} = A_G \otimes I_U + I_V \otimes A_H$ for any two graphs $G, H$ as above where, for a set $X$, the matrix $I_X$ denotes the matrix $(\delta_{xy})_{x,y \in X}$ (see [2, Theorem 2.21]). Similarly, the tensor product $x \otimes y$ of two vectors $x = (x_v)_{v \in V}$ and $y = (y_u)_{u \in U}$ is the vector $(x_v \cdot y_u)_{(v,u) \in V \times U}$, and we have $(A \otimes B)(x \otimes y) = Ax \otimes By$ for all such vectors $x, y$ and matrices $A, B$.

**Theorem 1** The product of semiharmonic graphs is semiharmonic.

**Proof** For $i = 1, 2$, let $G_i = (V_i, E_i)$ be a $\mu_i$-semiharmonic graph. Then, using the obvious fact that $j_{V_1} \otimes j_{V_2}$ coincides with the vector $j_{V_1 \times V_2}$, we have

$$(A_1 \otimes A_2)^3j_{V_1 \times V_2} = (A_1 \otimes A_2)(A_1 \otimes A_2)(A_1 j_{V_1} \otimes A_2 j_{V_2}) = A_1^3 j_{V_1} \otimes A_2^3 j_{V_2} = \mu_1 A_1 j_{V_1} \otimes \mu_2 A_2 j_{V_2} = \mu_1 \mu_2 (A_1 \otimes A_2) j_{V_1 \times V_2}$$

showing that $G_1 \times G_2$ is a $\mu_1 \mu_2$-semiharmonic graph. □

If $G_1$ and $G_2$ are both finite, connected, and strictly semiharmonic and thus bipartite, then $G_1 \times G_2$ has exactly two components which have the same non-zero eigenvalues, but they are not necessarily isomorphic (see [2] and [12]). Thus $G_1 \times G_2$ in general gives two new semiharmonic graphs.

**Theorem 2** For $i = 1, 2$, let $G_i = (V_i, E_i)$ be an $a_i, b_i$-semiregular graph and assume $a_1 b_1 = a_2 b_2$. Then the sum $G_1 + G_2$ is a $4c$-semiharmonic graph for $c := a_1 b_1 (= a_2 b_2)$.

**Proof** For $i = 1, 2$, put $j_i := j_{V_i}$ so that $A_i^2 j_i = c_j_i$ holds for $i = 1, 2$. Clearly, this implies for $j := j_{V_1 \times V_2} = j_1 \otimes j_2$:

$$(A_1 \otimes I_2 + I_1 \otimes A_2)^3 j = (A_1 \otimes I_2 + I_1 \otimes A_2)^2(A_1 j_1 \otimes j_2 + j_1 \otimes A_2 j_2) = (A_1^2 \otimes I_2 + 2A_1 \otimes A_2 + I_1 \otimes A_2^2)(A_1 j_1 \otimes j_2 + j_1 \otimes A_2 j_2) = A_1^3 j_1 \otimes j_2 + 3A_1^2 j_1 \otimes A_2 j_2 + 3A_1 j_1 \otimes A_2^2 j_2 + j_1 \otimes A_2^3 j_2 = cA_1 j_1 \otimes j_2 + 3c j_1 \otimes A_2 j_2 + 3A_1 j_1 \otimes c j_2 + j_1 \otimes cA_2 j_2 = 4c(A_1 \otimes I_2 + I_1 \otimes A_2)(j_1 \otimes j_2) = 4c(A_1 \otimes I_2 + I_1 \otimes A_2) j,$

showing that $G_1 \times G_2$ is a $4c$-semiharmonic graph. □

Note that the sum $G_1 + G_2$ of two semiregular graphs $G_1$ and $G_2$ need not be semiregular: e.g. the grid $P_3 + P_3$ is a bipartite graph in which one class contains four vertices of degree $2$ and one vertex of degree $4$.

The following construction is based on subdivision of edges of a regular multigraph.
Theorem 3 If $G$ is a $k$-regular multigraph and $G^*$ is the graph obtained by subdividing each edge of $G$ by three new vertices, then $G^*$ is a $(k+2)$-semiharmonic graph.

Proof In the graph $G^*$, let $V$ denote the set of vertices of $G$, $V'$ the set of new vertices adjacent to a vertex of $V$, and $V''$ the set of new vertices not adjacent to vertices of $V$. Thus, each edge of $G$ gives rise to a path $P_5$ in $G^*$ containing two vertices from $V$ and $V'$ each and one vertex from $V''$. We have:

i) If $u \in V$, then $d(u) = k$, $d_2(u) = 2k$, and $d_3(u) = k(k + 2)$,

ii) if $u \in V'$, then $d(u) = 2$, $d_2(u) = k + 2$, and $d_3(u) = 2k + 4$,

iii) if $u \in V''$, then $d(u) = 2$, $d_2(u) = 4$, and $d_3(u) = 2(k + 2)$.

We see that $d_3(u) = (k + 2)d(u)$ holds in each case, showing that $G^*$ is a $(k+2)$-semiharmonic graph. ■

The next and last construction enables us to get infinitely many new semiharmonic graphs starting from an almost semiregular graph: A graph $G = (V,E)$ is called almost semiregular if there exist constants $a$ and $c$ and a bipartition $V = V_1 \cup V_2$ of the set of its vertices with $E \subseteq \{(v_1,v_2) | v_1 \in V_1, v_2 \in V_2\}$ such that $d(v_1) = a$ and $d_2(v_1) = c$ holds for every vertex $v_1 \in V_1$ in which case $G$ will also be called almost $a,b$-semiregular for $b := c/a$. We will call the set $V_1$ the constant part of $G$ (even though in case $G$ is semiregular, this concept is not completely well defined because both classes $V_1$ and $V_2$ could then equally well be addressed as the constant part of $G$). Almost semiregular graphs are halfway between semiregular and semiharmonic graphs: Obviously, every $a,b$-semiregular graph is almost $a,b$-semiregular and every almost $a,b$-semiregular graph is $ab$-semiharmonic because we have $d_2(v_2) = a d(v_2)$ and $d_3(v_2) = a b d(v_2)$ for every $v_2 \in V_2$ as well as $d_3(v_1) = \sum_{v_2 \in N(v_1)} d_2(v_2) = \sum_{v_2 \in N(v_1)} a d(v_2) = a d_2(v_1) = a a b = a b d(v_1)$ for every $v_1 \in V_1$.

It will follow from the results presented in the next section and [4] that all semiharmonic graphs with up to two independent cycles are actually almost semiregular. However, there are semiharmonic graphs that are not almost semiregular— one such graph is shown in Fig. 1.

Our last construction is given in the following theorem.

Theorem 4 Let $G$ be an almost semiregular graph and $k \in \mathbb{N}$. Let $G^{+k}$ denote the graph obtained by attaching $k$ pendant vertices to each vertex of the constant part of $G$. Then, $G^{+k}$ is almost semiregular (thus semiharmonic).

Proof Let $V'$ be the constant part of $G$ and let $a, c$ be such that $d(v) = a$ and $d_2(v) = c$ hold for all $v \in V'$. After attaching $k$ pendant vertices to $v$, the new values of $d(v)$ and $d_2(v)$ become $d(v) = a + k$ and $d_2(v) = c + k$. ■
We see from the last construction that, in order to show that the number of finite semiharmonic graphs with a fixed positive cyclomatic number is infinite, it is enough to give an example of an almost semiregular graph with that cyclomatic number. Such an example may be obtained by taking copies of $C_4$, selecting one vertex from each copy and identifying all the selected vertices (see Fig. 2).

More generally, any bipartite graph $G = (V, E)$ with a symmetry group that acts transitively on one of the two parts of $G$ is necessarily almost semiregular (with that part as its constant part).

3 Bicyclic semiharmonic graphs

We define a vertex $v$ in a graph $G$ to be a bud if $v$ is not a leaf, yet all neighbors of $v$ except at most one are leaves, and a knob if $v$ has exactly two neighbors that are not leaves. Further, we define the graph $T_{a,b}$ for any $a, b \in \mathbb{N}$ to be the tree that contains a “central” vertex $v$ of degree $a$ all of whose neighbors are leaves in case $b = 1$ and buds of degree $b$ in case $b > 1$. Clearly, any such tree is $(a + b - 1)$-semiharmonic. A proof of the following lemma can be found in [4].

**Lemma 2** Let $G$ be a connected, semiharmonic graph.
(i) If $G$ contains a bud, it is a finite tree.

(ii) If $G$ is a finite tree, there exist $a, b \in \mathbb{N}$ with $G \cong T_{a,b}$.

For $a, k \in \mathbb{N}$ with $k \geq 2$, let $M_{a}^{2k} = (V_{a}^{2k}, E_{a}^{2k})$ denote the connected graph containing a cycle $(v_{0}, v_{1}, \ldots, v_{2k-1}, v_{2k} = v_{0})$ of length $2k$ of knobs, alternating of degree 2 and $2 + a$, i.e., with $d(v_{2i}) = 2$ and $d(v_{2i+1}) = 2 + a$ for $i = 0, 1, \ldots, k - 1$ while every neighbor of $v_{2i+1}$ except $v_{2i}$ and $v_{2i+2}$ (indices taken modulo 2$) is a leaf. Then $M_{a}^{2k}$ is a unicyclic $(4+a)$-semiharmonic graph, and every strictly semiharmonic unicyclic connected finite graph is of this form (cf. [4]).

**Lemma 3** Let $G = (V, E)$ be a strictly semiharmonic connected finite graph, and let $v_{0}, v_{1}, v_{2}, v_{3}, v_{4}$ be a path or a cycle (if $v_{0} = v_{4}$) of $G$ such that $v_{1}, v_{2}$ and $v_{3}$ are knobs. Let $a_{i} = d(v_{i}) - 2$ denote the number of leaves adjacent to $v_{i}$. Then one of the following three assertions holds:

(i) $a_{2} > 0$ and $G \cong M_{a_{2}}^{2k}$ for some $k \in \mathbb{N}$,

(ii) $a_{2} = 0$, $a_{1} = a_{3} > 0$, and $v_{0}, v_{4}$ are not adjacent to a leaf,

(iii) $a_{1} = a_{2} = a_{3} = 0$, and $G$ is obtained from a $d(v_{0})$-regular multigraph by subdividing every edge by three new vertices.

**Proof** Note first that $d(v_{i}) = a_{i} + 2$ holds for $i = 1, 2, 3$ and that, therefore,

\[
\begin{align*}
    d_{2}(v_{1}) & = a_{1} + a_{2} + 2 + d(v_{0}), \\
    d_{2}(v_{2}) & = a_{1} + a_{2} + a_{3} + 4, \\
    d_{2}(v_{3}) & = a_{2} + a_{3} + 2 + d(v_{4})
\end{align*}
\]

also hold. Further, since $G$ is a strictly semiharmonic, finite and connected graph, it is bipartite. We define $V_{1}$ and $V_{2}$ to be the bipartite classes containing $v_{1}$ and $v_{2}$, respectively, and we use Corollary 1 to conclude that there are rational numbers $\lambda_{1}$ and $\lambda_{2}$ such that $d_{2}(w_{i}) = \lambda_{i}d(w_{i})$ holds for every $i \in \{1, 2\}$ and every vertex $w_{i} \in V_{i}$.

In particular, we have

\[
\lambda_{2} = d_{2}(v_{2})/d(v_{2}) = (a_{1} + a_{2} + a_{3} + 4)/(a_{2} + 2),
\]

and, similarly,

\[
\lambda_{1} = (a_{1} + a_{2} + 2 + d(v_{0}))/((a_{1} + 2) = (a_{3} + a_{2} + 2 + d(v_{4}))/((a_{3} + 2).
\]

Note further that $d_{2}(x) = d_{2}(x)/d_{1}(x) = d(y)$ holds for every leaf $x$ and its unique neighbor $y$. So, one has $d(y) = \lambda_{1}$ in case $y$ is a vertex in $V_{2}$ adjacent to a leaf, and $d(y) = \lambda_{2}$ in case $y$ is a vertex in $V_{1}$ adjacent to a leaf. In particular, if $a_{2} > 0$ holds, we have

\[
\lambda_{1} = d(v_{2}) = a_{2} + 2.
\]
and we have
\[ \lambda_2 = d(v_i) = a_i + 2 \]
in case \( i = 1 \) or \( i = 3 \) and \( a_i > 0 \). Thus, if \( a_1, a_2 > 0 \) would hold and \( x_2 \) would be a leaf adjacent to \( v_2 \), we would have
\[ d_3(x_2) = d_2(v_2) = a_1 + a_2 + a_3 + 4 \]
as well as
\[ d_3(x_2) = \lambda_1 \lambda_2 d(x_2) = (a_1 + 2)(a_2 + 2) \]
and therefore
\[ a_3 = a_1 a_2 + a_1 + a_2 > a_1 > 0. \]
Thus, the same argument, with the roles of \( a_1 \) and \( a_3 \) interchanged, would imply that also \( a_1 > a_3 \) must hold, a contradiction.

In case \( a_2 = 0 \) and \( a_i > 0 \) for some \( i \in \{1, 3\} \), we have
\[ a_1 + 2 + a_3 + 2 = d(v_1) + d(v_3) = d_2(v_2) = d_2(v) \lambda_2 = 2(a_i + 2) \]
implying \( a_1 = a_3 \).
In consequence, we have either \( a_2 > 0 \) and \( a_1 = a_3 = 0 \), or \( a_2 = 0 \) and \( a_1 = a_3 > 0 \), or \( a_1 = a_2 = a_3 = 0 \).
In the first case, we get
\[ \lambda_2 = \frac{d_2(v_2)}{d(v_2)} = \frac{a_2 + 4}{a_2 + 2} \]
as well as
\[ 2(a_2 + 2) = d(v_3) \cdot \lambda_1 = d_2(v_3) = a_2 + 2 + d(v_4) \]
and, hence, \( d(v_4) = 2 + a_2 \) as well as \( d_2(v_4) = \lambda_2 d(v_4) = a_2 + 4 \) which implies that \( v_4 \) is adjacent to \( v_3 \), to one more vertex \( v_5 \) of degree 2, and to \( a_2 \) leaves. Let \( v_6 \neq v_5 \) be the other neighbor of \( v_5 \). Then, the vertices \( v_2, v_3, v_4, v_5, v_6 \) fulfill the conditions of the theorem, and we conclude that \( v_6 \) is adjacent to \( v_5 \), to one more vertex of degree 2, and to \( a_2 \) leaves. By iterating this argument, we see that \( G \cong M_{2k}^a \) must hold for \( a := a_2 \) and the smallest \( k \in \mathbb{N} \) with \( k(2 + a) = \#V_{2k}^a \geq \#V \).

Next, if \( a_2 = 0 \) and \( a_1 = a_3 > 0 \) hold, we have \( \lambda_2 = a_1 + 2 \) and
\[ \lambda_1 = \frac{d_2(v_1)}{d(v_1)} = \frac{d(v_0) + a_1 + 2}{a_1 + 2}. \]
Further, if, say, \( v_0 \) were adjacent to a leaf \( y \), in which case \( \lambda_1 = \lambda_1 d(y) = d_2(y) = d(v_0) \) would hold, this would yield
\[ d(v_0) = \lambda_1 = \frac{d(v_0) + a_1 + 2}{a_1 + 2}. \]
and hence \(d(v_0) = 1 + 1/(a_1 + 1)\) in contradiction to the fact that \(d(v_0)\) is a natural number.

Finally, if \(a_1 = a_2 = a_3 = 0\) holds, we obtain \(\lambda_2 = d_2(v_2)/d(v_2) = 2\) and

\[
\frac{d(v_0) + 2}{2} = \frac{d_2(v_1)}{d(v_1)} = \frac{d_2(v_3)}{d(v_3)} = \frac{d(v_1) + 2}{2},
\]

implying \(d(v_0) = d(v_4) = 2\lambda_1 - 2\). Since \(G\) is strictly semiharmonic, \(\lambda_1 \neq 2\) and, hence,

\[d(v_0) = d(v_4) \neq 2\]

must hold implying in particular that neither \(v_0\) nor \(v_4\) can be adjacent to a leaf, since for such leaf \(x\) we would have \(2 = \lambda_2 = d_2(x)/d(x) = d(v_0) = d(v_4) \neq 2\).

In case \(d(v_0) = 1\), \(G\) must be isomorphic to the semiharmonic tree \(T_{2,1}\) in accordance with Assertion (iii). Otherwise, we must have \(d(v_0) = d(v_4) \geq 3\) as well as \(d(v) \geq 2\) for all \(v \in V_2\) as \(d(v) = 1\) for some \(v \in V_2\) with neighbor \(u\) would imply that \(u\) must be a bud in view of \(d(u) = d_2(u) = \lambda_2 d(v) = 2\) implying in turn that \(G\) must be of the form \(T_{a,b}\) for some \(a,b \in \mathbb{N}\) and, hence, isomorphic to \(T_{2,1}\). Further, every neighbor \(w_1\) of \(v_0\) (or \(v_4\)) must have degree 2 in view of \(d_2(v_0) = \lambda_2 d(v_0) = 2 d(v_0)\) and the fact that \(d(w) \geq 2\) holds for every neighbor \(w\) of \(v_0\) (or \(v_4\)), while the other neighbor \(w_2\) of any such neighbor \(w_1\) of \(v_0\) must also have degree 2 in view of

\[d(w_2) + d(v_0) = d_2(w_1) = \lambda_1 d(w_1) = 2\lambda_1 = d(v_0) + 2\]

Further, we must have

\[d(w_3) + d(w_1) = d_2(w_2) = \lambda_2 d(w_2) = 4\]

and, hence, \(d(w_3) = 2\) for the other neighbor \(w_3 \neq w_1\) of \(w_2\). Thus, applying the argument used above for \(v_0,\ldots,v_4\) once again, but now for the vertices \(v_0, w_1, w_2, w_3, w_4\) where \(w_4\) is the other neighbor \(w_4 \neq w_2\) of \(w_3\), we get \(d(w_4) = d(v_0)\) for that neighbor \(w_4\). Since \(w_1\) was assumed to be an arbitrary neighbor of \(v_0\) distinct from \(v_1\), every vertex at distance 1, 2, or 3 from \(v_0\) has degree 2, and every vertex at distance 4 from \(v_0\) has degree \(d(v_0)\). Repeating this argument in the same way, one can easily establish Assertion (iii).

**Theorem 5** A finite connected graph \(G\) is semiharmonic and bicyclic if and only if it belongs to one of the families depicted in Fig. 3 where, for any fixed non-negative integer \(k\), three small rays mean that exactly \(k\) leaves are attached to each such decorated vertex.

**Proof** Let \(G\) be a connected semiharmonic bicyclic graph. Since there are no harmonic bicyclic graphs (see [1]), \(G\) must be strictly semiharmonic. Let the skeleton \(sk(G)\) of a graph \(G\) be the graph obtained by removing all leaves of \(G\). By Lemma 2, \(G\) contains no buds and, thus, the skeleton \(sk(G)\) contains no
Figure 3: Families of semiharmonic bicyclic graphs.

Figure 4: Possible cycle structures for a bicyclic graph.

Figure 5: Possible cycle structures for a bicyclic semiharmonic graph.
leaves. Since \( \text{sk}(G) \) is also a bicyclic graph, it must have one of the forms shown in Fig. 4, where each arc represents a path of arbitrary length.

Internal vertices on these arcs are knobs of \( G \). If there were more than three knobs on any of these arcs, we could apply Lemma 3 to each three consecutive knobs on such an arc to deduce that Assertion (iii) must hold for all such triples of knobs. But then \( G \) could be obtained from a 2-regular multigraph by subdividing every edge by three new vertices, i.e. \( G \) would be a cycle itself — a contradiction.

Thus, each of the arcs contains at most three internal vertices. Since \( G \), being a strictly semiharmonic graph, must be bipartite, the length of all its cycles must be even. It follows that only eleven distinct possibilities exist for \( \text{sk}(G) \) (see Figure 5).

For any vertex \( v \) of \( \text{sk}(G) \), let \( l(v) \) denote the number of leaves that are in \( G \) adjacent to \( v \), and let \( ad(v) = \frac{d_2(v)}{d_2(v)} \) denote, as before, the average degree of the neighbors of \( v \) in \( G \). We will now go through all eleven cases for \( \text{sk}(G) \), and for each case we will either derive a contradiction or characterize all semiharmonic graphs with that skeleton.

![Figure 6: The graph \( G_1 \)](image)

**Case 1:** \( \text{sk}(G) \cong G_1 \). By Lemma 3, leaves can only be attached to \( u_1, u_2, w_1, w_2 \), and we have \( l(u_1) = l(u_2) \) and \( l(w_1) = l(w_2) \). Further, \( ad(t) = ad(x) \) implies \( l(u_1) = l(w_1) \). If \( l(u_1) = l(u_2) = l(w_1) = l(w_2) \) holds then the graph is \( (l(a)+6) \)-semiharmonic.

![Figure 7: The graph \( G_2 \)](image)
Case 2: $\text{sk}(G) \cong G_2$. By Lemma 3, leaves can only be attached to $u_1, u_2, x_1, x_2$, and we have $l(u_1) = l(u_2)$ and $l(x_1) = l(x_2)$. Further, $l(u_1) + 2 = \text{ad}(t) = \text{ad}(v) = \frac{2(l(u_1) + 7)}{3}$ implies $l(u_1) = 1$. By symmetry, $l(x_1) = 1$ must hold as well, but the obtained graph is not semiharmonic.

Figure 8: The graph $G_3$

Case 3: $\text{sk}(G) \cong G_3$. By Lemma 3, there are no leaves attached to $s, u, w, y$, and $l(t_1) = l(t_2)$ and $l(x_1) = l(x_2)$ holds. From $\text{ad}(s) = \text{ad}(y)$, we conclude $l(t_1) = l(x_1)$. Further,

$$l(t_1) + 2 = \text{ad}(s) = \text{ad}(u) = \frac{1}{3} (2(l(t_1) + 2) + l(v) + 2)$$

implies $l(v) = l(t_1)$. Thus, we obtain $\text{ad}(v) = \frac{l(t_1) + 6}{l(t_1) + 2} \neq \frac{l(t_1) + 5}{l(t_1) + 2} = \text{ad}(t_1)$, implying that no semiharmonic graph $G$ with $\text{sk}(G) \cong G_1$ can exist.

Figure 9: The graph $G_4$
Case 4: $\text{sk}(G) \cong G_4$.

By Lemma 3, there are no leaves attached to $s, u, x$, and $z$, and $l(t_1) = l(t_2)$ and $l(y_1) = l(y_2)$ hold. Since $s$ and $y_1$ are contained in the same bipartite class of $V$,

$$l(t_1) + 2 = ad(s) = ad(y_1) = \frac{l(y_1) + 5}{l(y_1) + 2}$$

must hold implying that $(l(t_1) + 1)(l(y_1) + 2) = 3$. Since $l(t_1), l(y_1)$ are non-negative integers, $l(t_1) = 0$ and $l(y_1) = 1$ must hold, but in view of Lemma 3, $l(t_1) = 0$ implies that $G$ is obtained from a regular multigraph by subdividing every edge by three vertices, a contradiction to the fact that the distance between the vertices $u$ and $x$, both of degree 3, is only 3.

![Figure 10: The graph $G_5$](image)

Case 5: $\text{sk}(G) \cong G_5$. By Lemma 3, leaves can only be attached to $s_1, s_2, u, w, y_1, y_2$, and we have $l(s_1) = l(s_2)$ and $l(u) = l(w)$ and $l(y_1) = l(y_2)$. Since $r, v, z$ are all in the same bipartite class of $G_5$, we have $l(s_1) = l(u) = l(y_1)$. The obtained graph is $(l(a) + 5)$-semiharmonic.

![Figure 11: The graph $G_6$](image)
Case 6: sk(G) ≅ G₆. First we assume \( l(u) = l(x) = 0 \). Since \( G₆ \) is not semiharmonic, we can assume without loss of generality that \( l(v₂), l(w₁) \leq l(v₁) > 0 \) holds. Let \( y \) be a leaf adjacent to \( v₁ \). Then \( ad(u) = ad(y) = l(v₁) + 2 \) must hold, but that implies \( l(v₁) = l(v₂) = 1 \) in view of \( d(x) = 3 \) and \( l(v₂) \leq l(v₁) \). Since \( y, w₁, w₂ \) are in the same bipartite class of \( G \), we have \( 3 = ad(y) = ad(w₁) = ad(w₂) \) and thus \( l(w₁) = l(w₂) = 0 \). However, the obtained graph is not semiharmonic.

Now we assume that there is a leaf attached to \( u \) or \( x \), say \( l(x) > 0 \). Let \( z \) be a leaf adjacent to \( x \). If there were some \( i \in \{1, 2\} \) with \( l(vᵢ) > l(x) + 1 \), we would obtain \( ad(y) > ad(z) \) for any leaf \( y \) adjacent to \( vᵢ \), a contradiction. Thus, we have \( l(vᵢ) \leq l(x) + 1 \) for \( i = 1, 2 \), and it follows from

\[
l(x) + 3 = ad(z) = ad(u) = \frac{l(v₁) + l(v₂) + 4 + l(u) + l(x) + 3}{l(u) + 3}
\]

that \( (l(u) + 2)(l(x) + 2) = l(v₁) + l(v₂) + 2 \) must hold implying \( l(u) = 0 \) and \( l(vᵢ) = l(x) + 1 \) for \( i = 1, 2 \) in view of \( l(vᵢ) \leq l(x) + 1 \). By

\[
\frac{2(l(x) + 3) + l(wᵢ)}{2 + l(wᵢ)} = ad(wᵢ) = ad(z) = l(x) + 3,
\]

we obtain \( l(wᵢ)(l(x) + 2) = 0 \) implying \( l(wᵢ) = 0 \) for \( i = 1, 2 \). Hence, we have \( ad(x) = \frac{l(x) + 7}{l(x) + 3} \neq \frac{l(x) + 6}{l(x) + 3} = ad(v₁) \), a contradiction.

![Figure 12: The graph G₇](image)

Case 7: sk(G) ≅ G₇. First assume that \( l(u) = l(x) = 0 \). Since \( G₇ \) is not semiharmonic, we can assume without loss of generality that \( l(v₂), l(v₃), l(w₁) \leq l(v₁) > 0 \) holds. Let \( y \) be a leaf adjacent to \( v₁ \). Then \( ad(u) = ad(y) = l(v₁) + 2 \) must hold which implies \( l(v₁) = l(v₂) = l(v₃) \) in view of \( l(v₂), l(v₃) \leq l(v₁) \). Since, for \( i = 1, 2, 3 \), the vertices \( y, wᵢ \) are in the same bipartite class of \( V \), we conclude that

\[
\frac{l(wᵢ) + l(v₁) + 2 + 3}{l(wᵢ) + 2} = ad(wᵢ) = ad(y) = l(v₁) + 2,
\]

and, hence, \( (l(v₁) + 1)(l(wᵢ) + 1) = 2 \) must hold. Thus, we must have \( l(v₁) = 1 \) and \( l(wᵢ) = 0 \), and the obtained graph is 6-semiharmonic.
Now assume that there is a leaf attached to \( u \) or \( x \), say \( l(x) > 0 \). Let \( z \) be a leaf adjacent to \( x \). If there were some \( i \in \{1, 2, 3\} \) with \( l(v_i) > l(x) + 1 \), we would obtain \( \text{ad}(y) > \text{ad}(z) \) for any leaf \( y \) adjacent to \( v_i \), a contradiction. Thus, we have \( l(v_i) \leq l(x) + 1 \) for \( i = 1, 2, 3 \), and it follows from

\[
l(x) + 3 = \text{ad}(z) = \text{ad}(u) = \frac{l(v_1) + l(v_2) + l(v_3) + 6 + l(u)}{l(u) + 3}
\]

that \( (l(u) + 3)(l(x) + 2) = l(v_1) + l(v_2) + l(v_3) + 3 \) must hold implying \( l(u) = 0 \) and \( l(v_i) = l(x) + 1 \) for \( i = 1, 2, 3 \) in view of \( l(v_i) \leq l(x) + 1 \). By

\[
\frac{2(l(x) + 3) + l(w_i)}{2 + l(w_i)} = \text{ad}(w_i) = \text{ad}(z) = l(x) + 3,
\]

we obtain \( l(w_i)(l(x) + 2) = 0 \) implying \( l(w_i) = 0 \) for \( i = 1, 2, 3 \). The obtained graph is \( (l(x) + 6)\)-semiharmonic for every \( l(x) \in \mathbb{N} \).

![Figure 13: The graph \( G_8 \)](image)

**Case 8:** \( \text{sk}(G) \cong G_8 \). Since \( G \) is semiharmonic we must have \( \text{ad}(u) = \text{ad}(w) \). This implies \( l(u) = l(w) \) in view of

\[
\text{ad}(u) = \frac{l(u) + l(v_1) + l(v_2) + l(v_3) + 6}{l(u) + 3} = 1 + \frac{l(v_1) + l(v_2) + l(v_3) + 3}{l(u) + 3}
\]

and

\[
\text{ad}(w) = \frac{l(w) + l(v_1) + l(v_2) + l(v_3) + 6}{l(w) + 3} = 1 + \frac{l(v_1) + l(v_2) + l(v_3) + 3}{l(w) + 3}.
\]

Further, we must have \( \text{ad}(v_1) = \text{ad}(v_2) = \text{ad}(v_3) \). This implies \( l(v_1) = l(v_2) = l(v_3) \) in view of

\[
\text{ad}(v_i) = \frac{l(u) + l(w) + 6 + l(v_i)}{l(v_i) + 2} = 1 + \frac{l(u) + l(w) + 4}{l(v_i) + 2}
\]

for \( i \in \{1, 2, 3\} \).

If \( l(u) = l(v_1) = 0 \) then \( G \) is a 6-semiharmonic graph.
If \( l(u) > 0 \) then let \( y \) be a leaf adjacent to \( u \). We have
\[
l(u) + 3 = ad(y) = ad(v_1) = \frac{2(l(u) + 3) + l(v_1)}{2 + l(v_1)} = l(u) + 3 - \frac{(l(u) + 2)l(v_1)}{l(v_1) + 2}
\]
which implies \( l(v_1) = 0 \). In that case \( G \) is an \((l(u) + 6)\)-semiharmonic graph.

If \( l(v_1) > 0 \) then let \( z \) be a leaf adjacent to \( v_1 \). We have
\[
l(v_1) + 2 = ad(z) = ad(u) = \frac{3(l(v_1) + 2) + l(u)}{3 + l(u)} = l(v_1) + 2 - \frac{(l(v_1) + 1)l(u)}{l(u) + 3}
\]
which implies \( l(u) = 0 \). In that case \( G \) is an \((l(v_1) + 6)\)-semiharmonic graph.

**Case 9:** \( sk(G) \cong G_9 \). By Lemma 3, leaves can only be attached to \( v_1, v_2, w, y \), and we have \( l(w) = l(y) \). Further, we have
\[
1 + \frac{4}{l(v_1) + 2} = \frac{l(v_1) + 6}{l(v_1) + 2} = ad(v_1) = ad(v_2) = \frac{l(v_2) + 6}{l(v_2) + 2} = 1 + \frac{4}{l(v_2) + 2},
\]
implying \( l(v_1) = l(v_2) \). Moreover, \( l(v_1) = l(w) \) holds in view of
\[
l(w) + 2 = ad(x) = ad(u) = \frac{2(l(v_1) + 2) + l(w) + 2}{3} = \frac{2l(v_1) + l(w)}{3} + 2.
\]
The obtained graph can not be semiharmonic, since we have
\[
ad(v_1) = \frac{l(v_1) + 6}{l(v_1) + 2} \neq \frac{l(v_1) + 5}{l(v_1) + 2} = ad(w).
\]

**Case 10:** \( sk(G) \cong G_{10} \). By Lemma 3, leaves can only be attached to \( v, w_1, w_2, y_1, y_2 \), and we have \( l(w_1) = l(y_1) \) and \( l(w_2) = l(y_2) \). Further, we have \( l(w_1) = l(w_2) \) in view of \( l(w_1) + 2 = ad(x_1) = ad(x_2) = l(w_2) + 2 \), and we have \( l(v) = l(w_1) \) in view of
\[
\frac{2l(w_1) + l(v)}{3} + 2 = ad(u) = ad(x_1) = l(w_1) + 2.
\]
The obtained graph is not semiharmonic since we have

\[ ad(v) = \frac{l(v) + 6}{l(v) + 2} \neq \frac{l(v) + 5}{l(v) + 2} = ad(w_1). \]

**Case 11:** \( sk(G) \cong G_{11} \). By Lemma 3, leaves can only be attached to \( u_1, u_2, u_3, w_1, w_2, w_3 \), and we have \( l(u_i) = l(w_i) \) for \( i \in \{1, 2, 3\} \). For \( i, j \in \{1, 2, 3\} \), we have \( l(u_i) + 2 = ad(v_i) = ad(v_j) = l(u_2) + 2 \), thus \( l(u_1) = l(u_2) = l(u_3) \) holds. The resulting graph is \((l(u_1) + 5)\)-semiharmonic.

After having considered all eleven cases in Fig. 5, we get that there are exactly six infinite families of semiharmonic bicyclic graphs all of which are depicted in Fig. 3.

In [5], we have investigated more generally the skeleta of finite semiharmonic graphs and shown that, even though there are infinitely many (isomorphism classes of) finite connected semiharmonic graphs with given cyclomatic number \( c \) for any \( c \geq 0 \), defining two such graphs to be \textit{sk-equivalent} if their skeleta are isomorphic, the number of \textit{sk-equivalence} classes of finite connected semiharmonic graphs with given cyclomatic number \( c \) is finite for all \( c \geq 2 \).
References


