Research problems from
the Aveiro Workshop on Graph Spectra

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Abstract

This is a collection of open problems presented at the Aveiro Workshop on Graph Spectra held at the University of Aveiro, Portugal from April 10-12, 2006.

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These problems were presented at the problem session of the Aveiro Workshop on Graph Spectra at the University of Aveiro, April 10–12, 2006. They, and a few other, appear at the web page Open Problems in Spectral Graph Theory, which is available at http://www.sgt.pep.ufrj.br and mirrored at http://160.99.54.70/openproblems/index.php.

Problem AWGS.1. The maximum clique and the signless Laplacian
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Let $Q(G) = D + A$ be the signless Laplacian matrix of a graph $G$, where $D$ is the degree, and $A$ is the adjacency matrix of $G$. Computer experiments show that for certain graphs the size of the maximum clique $\omega(G)$ is equal to $\lambda_{\min}(Q(G)) + 2$. However, there are graphs for which $\omega(G)$ is larger than or smaller than $\lambda_{\min}(Q(G)) + 2$.

Problem. Find conditions on $G$ which yield equality $\omega(G) = \lambda_{\min}(Q(G)) + 2$ or permit a bound on $\omega(G)$ in terms of $\lambda_{\min}(G)$.

Problem AWGS.2. Integral graphs
Krzysztof Zwierzyński
This is actually a set of three problems regarding integral graphs. A graph is integral if all the eigenvalues of its adjacency matrix are integers. The search for integral graphs started already in 1970s and continues to this day. During the session, Horst Sachs supported the topic of integral graphs by pointing out that these are exactly the graphs whose characteristic polynomials are decomposable into linear factors over the field of rationals.

So far, integral graphs have been found or characterized in various graph classes. Using Brendan McKay’s program \texttt{geng} for generating graphs, nowadays it is easy to see that there are exactly 263 connected integral graphs on up to 11 vertices (see [2, 3]). On more than eleven vertices there are (still) too many graphs for exhaustive generation. Using the evolutionary techniques (see [3, 4]) it is possible to construct (presumably all) 325 connected integral graphs on 12 vertices and 541 connected integral graphs on 13 vertices. However, there is no proof that these are all such graphs. Thus,

\textbf{Problem A.} What are the exact numbers of connected integral graphs on 12 and 13 vertices?

Related to the topic of generating integral graphs by computer is the following, more technical

\textbf{Problem B.} Can one determine the size of a bipartition given only the spectrum of a connected, bipartite graph?

Note that, should the answer to the above problem be positive, there may be more than one possible bipartition size. Observe the following construction by Willem Haemers (private communication).

Let $J_{a,b}$ denote the $a \times b$ all-one matrix and let $O_{a,b}$ denote the $a \times b$ all-zero matrix. Put

$\begin{align*}
N &= \begin{bmatrix}
J_{a+1,a} & O_{a+1,a} & J_{a+1,1} \\
O_{a+3,a} & J_{a+3,a} & J_{a+3,1}
\end{bmatrix} \\
N' &= \begin{bmatrix}
J_{a,a+1} & O_{a,a+1} & J_{a,1} \\
O_{a+2,a+1} & J_{a+2,a+1} & J_{a+2,1}
\end{bmatrix}.
\end{align*}$
Then the bipartite incidence graphs $G$ and $G'$ of $N$ and $N'$, respectively, are cospectral. Both graphs have $4a+5$ vertices and $2(a+1)(a+2)$ edges. Further, both graphs have rank 4, so it suffices to check the nonzero eigenvalues only, which are also the nonzero eigenvalues of the $5 \times 5$ quotient matrices consisting of the row sums of the blocks. However, $G$ has $2a + 4$ vertices in one part and $2a + 1$ vertices in another part, while $G'$ has $2a + 3$ vertices in one part and $2a + 2$ vertices in another part.

Haemers claims to have found, together with Edwin van Dam, more examples including an infinite family that is also cospectral with respect to the complement (unlike the example above). In all these cases the sizes of the bipartition classes differ only by 1. However, he expects that, by the same method, one could find examples where the bipartition sizes differ by more than 1.

If a graph $G$ is regular, then the spectra of its Laplacian matrix $L(G) = D - A$ and the signless Laplacian matrix $Q(G) = D + A$ are easily determined from its adjacency spectrum. Moreover, if a regular graph $G$ is integral, then its Laplacian spectrum and signless Laplacian spectrum also consist of integers.

**Problem C.** Determine all connected, nonregular graphs that are integral, Laplacian integral and signless Laplacian integral.

See the paper by Steve Kirkland in this issue.

**Problem AWGS.3.** Spectral properties of block matrices

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Motivation for the following problem may be found in [6, 15]. The matrices

\[
\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \text{and} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]

are called *Pauli matrices*. Let $I$ be the identity matrix. Given a graph $G = (V, E)$, let us construct a block-matrix $S(G)$ with $i\bar{j}$-th block defined as
follows:

\[ S(G)_{i,j} = \begin{cases} 
\sigma_x, & \text{if } i = j; \\
\sigma_z, & \text{if } \{i, j\} \in E(G); \\
I, & \text{if } \{i, j\} \notin E(G). 
\end{cases} \]

From the block-columns of \( S(G) \) we construct the matrices

\[
S(G)_1 = S(G)_{1,1} \otimes S(G)_{1,2} \otimes \cdots \otimes S_{1,n}, \\
S(G)_2 = S(G)_{2,1} \otimes S(G)_{2,2} \otimes \cdots \otimes S_{2,n}, \\
\vdots \\
S(G)_n = S(G)_{n,1} \otimes S(G)_{n,2} \otimes \cdots \otimes S_{n,n}.
\]

It can be shown that these matrices all commute. The graph state associated to the graph \( G \) is defined to be the common eigenvector of the matrices \( S(G)_1, S(G)_2, \ldots, S(G)_n \) with eigenvalue 1.

**Problem.** Is there a relation between the graph state and the spectral properties of the adjacency matrix of a graph?

**Problem AWGS.4.** Cospectrality measure

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Let \( G_n \) and \( G'_n \) be two nonisomorphic graphs on \( n \) vertices with spectra

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad \text{and} \quad \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n,
\]

respectively. Define the distance between the spectra of \( G_n \) and \( G'_n \) as

\[
\lambda(G_n, G'_n) = \sum_{i=1}^{n} (\lambda_i - \lambda'_i)^2 \quad \text{(or use } \sum_{i=1}^{n} |\lambda_i - \lambda'_i|).\]

Let \( \epsilon \) be a nonnegative number. Graphs \( G_n \) and \( G'_n \) are \( \epsilon \)-cospectral if \( \lambda(G_n, G'_n) \leq \epsilon \). Thus, \( G_n \) and \( G'_n \) are 0-cospectral if and only if \( G_n \) and \( G'_n \) are cospectral. Define the cospectrality of \( G_n \) by

\[
\text{cs}(G_n) = \min \{ \lambda(G_n, G'_n) : G'_n \text{ not isomorphic to } G_n \}.\]
Thus $cs(G_n) = 0$ if and only if $G_n$ has a cospectral mate. Let

$$cs_n = \max\{cs(G_n) : G_n \text{ a graph on } n \text{ vertices}\}.$$ 

This function measures how far apart the spectrum of a graph with $n$ vertices can be from the spectrum of any other graph with $n$ vertices.

**Problem A.** Investigate $cs(G_n)$ for special classes of graphs.

**Problem B.** Find a good upper bound on $cs_n$.

**Problem AWGS.5.** Laplacian spectra consisting of all distinct integers

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In the course of investigating the graphs whose Laplacian spectra consist of all distinct integers (see [10]), the realizability (or lack thereof) of the set

$$\{0, 1, 2, \ldots, n-2, n-1\}$$

as the Laplacian spectrum of some graph is central. In particular, if there is no such graph, it then follows that for each collection $S$ of $n$ distinct integers, there is at most one graph on $n$ vertices whose Laplacian spectrum is $S$. Characterizations of the admissible sets, and constructions of the graphs that attain them as their spectra, will also follow.

**Problem.** Prove that there is no graph on $n$ vertices whose Laplacian spectrum is given by $\{0, 1, \ldots, n-2, n-1\}$.

If we suppose that $G$ is a (hypothetical?) graph on $n$ vertices with Laplacian spectrum $\{0, 1, 2, \ldots, n-2, n-1\}$, then we can say following:

a) $n \equiv 0$ or $1 \pmod{4}$;

b) $(n - 1)!$ is divisible by $n$;

c) $n \geq 12$;

d) $G$ has $\frac{n(n-1)}{4}$ edges;
e) if $G$ has degree sequence $d_1, \ldots, d_n$, then $2 \leq d_i \leq n - 3$ for all $i$, and
$$
\sum d_i^2 = n(n - 1)(n - 2)/3.
$$

**Problem AWGS.6.** The maximum spread
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The spectral spread of a graph is the difference $\lambda_1 - \lambda_n$ between the largest and the smallest eigenvalue of its adjacency matrix.

A complete split graph $CS(n, q)$, $q \leq n$, is a graph on $n$ vertices consisting of a clique on $q$ vertices and a stable set on the remaining $n - q$ vertices in which each vertex of the clique is adjacent to each vertex of the stable set.

**Conjecture.** For each $n$ the maximum value of the spectral spread of a graph on $n$ vertices is obtained uniquely for a complete split graph $CS(n, \lfloor \frac{2n}{3} \rfloor)$.

Note that the spread of the adjacency matrix of $CS(n, \lfloor \frac{2n}{3} \rfloor)$ is equal to
$$
\sqrt{\lfloor (4/3)(n^2 - n + 1) \rfloor}.
$$
Gregory et al. [13] verified this conjecture by computer for all graphs up to 9 vertices. The same conjecture has been posed independently in [1] after some computer experiments which provide evidence that the conjecture is also true for higher values of $n$.

**Problem AWGS.7.** The maximum irregularity
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The irregularity of a graph is the difference between its index and its average vertex degree [8]. A pineapple $PA(n, q)$, $q \leq n$, is a graph on $n$ vertices consisting of a clique on $q$ vertices and a stable set on the remaining
$n - q$ vertices in which each vertex of the stable set is adjacent to the same, unique vertex of the clique.

**Conjecture.**[1] The most irregular connected graph on $n$ vertices ($n \geq 10$) is a pineapple $PA(n, q)$ in which the clique size $q$ is equal to $\left\lceil \frac{n}{2} \right\rceil + 1$.

Using known bounds on the index of a graph, the irregularity may be bounded as

$$\lambda_1 - \overline{d} \leq \frac{n}{4} - 1 + \frac{1}{n},$$

which is close (but not equal) to the irregularity of pineapple $PA(n, \left\lceil \frac{n}{2} \right\rceil + 1)$. Further, we know only that the connected graph of maximal irregularity has a stepwise adjacency matrix.

**Problem AWGS.8.** Nordhaus-Gaddum for index of graphs

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**Conjecture.**[1] The maximal graphs on $n$ vertices for the function $\lambda_1(G) + \lambda_1(\overline{G})$ are the complete split graphs $CS(n, q)$ with the clique size equal or close to $\frac{n}{3}$.

More precisely, for any simple graph $G$ with complement $\overline{G}$ on $n$ vertices we have

$$\lambda_1(G) + \lambda_1(\overline{G}) \leq \frac{4n - 5}{3} - \begin{cases} f_1(n), & \text{if } n \equiv 1 \pmod{3}, \\ 0, & \text{if } n \equiv 2 \pmod{3}, \\ f_2(n), & \text{if } n \equiv 0 \pmod{3}. \end{cases}$$

where

$$f_1(n) = \frac{3n - 2 - \sqrt{9n^2 - 12n + 12}}{6}, \quad f_2(n) = \frac{3n - 1 - \sqrt{9n^2 - 6n + 9}}{6}.$$

Let $\Delta, \delta$ be the maximum and minimum vertex degree of a graph $G$. It is easy to see that if $\Delta - \delta \leq \frac{n-2}{3}$, then

$$\lambda_1(G) + \lambda_1(\overline{G}) \leq \frac{4n - 5}{3}.$$
However, not much is known in the general case.

**Problem AWGS.9.** A classic problem  
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In this problem, we are (once again) interested to find the maximum index of graphs with given numbers of vertices and edges.

A *fanned pineapple FPA*$_i(n, q, t)$ of type $i$, $i = 1, 2$, $n \geq q \geq t$, is a graph on $n$ vertices obtained from a pineapple $PA(n, q)$ by connecting a vertex from the stable set by edges to $t$ vertices of the clique when $i = 1$ ($0 \leq t \leq q - 2$), and to $t$ vertices of the stable set when $i = 2$ ($0 \leq t < n - q$).

Let $H(n, n + k)$ be the set of all connected graphs with $n$ vertices and $n + k$ edges, and let (according to [9])

\[
G_{n,k} = FPA_1(n, d, t), \quad \text{where } k = \left(\frac{d - 1}{2}\right) + t - 1, \quad d - 2 \geq t \geq 0, \\
H_{n,k} = FPA_2(n, 1, k + 1).
\]

**Conjecture.**[1] Let $k \geq 3$, and let $G$ be a graph of maximal index in $H(n, n + k)$.

(i) If $n < f(k)$, then $G \cong G_{n,k}$;

(ii) If $n = f(k)$, then $G \cong G_{n,k}$ or $G \cong H_{n,k}$;

(iii) If $n > f(k)$, then $G \cong H_{n,k}$.

(For the definition of function $f(k)$, see the full paper [1].)

This problem is one of great interest in spectral graph theory and has a long history. So far, the above conjecture has been proved for $k = 3, 4, 5, 6$ and $k = \binom{m}{2} - 1$, $m \geq 4$ (see [5] in latter case). Further, it is known that the maximal graph is $H_{n,k}$ for given $k$ and sufficiently large $n$ [9]. Other than that, it is known only that maximal graphs have a *stepwise* adjacency matrix, i.e., they are exactly the nested split graphs, characterized by the property that they do not contain any of $P_4$, $2K_2$ and $C_4$ as an induced subgraph.
**Problem AWGS.10.** The Spectrum of the (3, 6)-Cage
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The chemistry of carbon cages (fullerenes) and related compounds has aroused much interest in the spectral theory of embedded graphs (polyhedra).

A \((p, q)\)-cage is (the graph of) a simple (i.e., trivalent) polyhedron each face of which is a \(p\)-gon or a \(q\)-gon. The \((5, 6)\)-cages correspond to the spherical fullerenes. The \((3, 6)\)-cages are closely related to the toroidal fullerenes (trivalent hexagonal tessellations of the torus); their structure can easily and intuitively be described by only three parameters (see [11]).

By Euler’s formula, every \((3, 6)\)-cage has precisely 4 triangles (and an even number of hexagons); the smallest is the tetrahedron with \(\{3, -1, -1, -1\}\) as its spectrum. In 1995 Patrick W. Fowler (see [11, p. 159]) posed the following

**Conjecture.** The spectrum of any \((3, 6)\)-cage with \(h\) hexagons (thus with \(2h + 4\) vertices) has the form

\[
\{3, -1, -1, -1; \lambda_1, \lambda_2, \ldots, \lambda_h; -\lambda_1, -\lambda_2, \ldots, -\lambda_h\}.
\]

Earlier, we (P.W. Fowler, P.E. John, H.S., see [11]) have found some reductions which enabled the conjecture to be proved for many infinite subclasses of the class of all \((3, 6)\)-cages but, in its generality, the problem is still open.

**Note added in proof:** We (P.E. John and H.S.) have just solved the problem (in fact, we did a bit more) but it remains a challenge to find a brief and elegant proof.

**Problem AWGS.11.** The minimum rank of matrices
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The set of symmetric matrices associated to a simple graph \(G\) is

\[
S(G) = \{A: A = A^T \text{ and for } i \neq j, a_{ij} \neq 0 \iff ij \in E\}.
\]
Note the diagonal of $A$ is ignored. The minimum rank of $G$ is

$$mr(G) = \min_{B \in S(G)} \text{rank}(B).$$

**Problem A.** What techniques from spectral graph theory would be useful here?

There are easy-to-use algorithms for computing the minimum rank of trees, and there are results allowing the computation of the minimum rank of a graph with a cut vertex from the minimum ranks of the pieces. Graphs having minimum rank 1, 2 and $|V(G)| - 1$ are known.

**Problem B.** For what families of matrices would the minimum rank be of interest to spectral graph theorists?

One example (motivating Problem B) is the following (see [14]). The largest number $k$ for which an independent set of $k$ vertices exists is called the vertex independence number of $G$ and denoted by $\alpha(G)$. The clique covering number of $G$, denoted by $\theta(G)$, is the smallest number of cliques in a clique covering of $G$. $\eta(G)$ is defined to be the smallest rank of any $n \times n$ (not necessarily symmetric) matrix $M$ (over any field), which satisfies $m_{ii} \neq 0$ for $i = 1, \ldots, n$ and $m_{ij} = 0$, if $v_i$ and $v_j$ are distinct nonadjacent vertices. It is known that

$$\alpha(G) \leq \eta(G) \leq \theta(G).$$

Note that $\eta$ is not comparable to $mr$.

**Problem AWGS.12** Regularity property of distinct eigenvalues

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Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$ and denote by $N_G(v)$ the open neighborhood of $v$ in $G$ (that is, $N_G(v) = \{u \in V(G) : uv \in E(G)\}$).

A $(k, \tau)$-regular set in a graph $G$ is a subset $S \subseteq V(G)$ such that

$$(\forall v \in V(G)) \quad |N_G(v) \cap S| = \begin{cases} k, & \text{if } v \in S; \\ \tau, & \text{otherwise}. \end{cases}$$
For a $p$-regular graph $G$, we say that $V(G)$ is $(p,0)$-regular set.

It is known [16] that if $G$ is a regular graph with a $(k,\tau)$-regular set, then $k - \tau$ is an eigenvalue of its adjacency matrix. There are even regular graphs such that for each distinct eigenvalue $\lambda$ there exists a $(k,\tau)$-regular set with $\lambda = k - \tau$ (note that the $(p,0)$-regular set $V(G)$ corresponds to the largest eigenvalue $p$ of $G$).

For example, in Petersen graph $P$ (see Fig. 1) the index $\lambda = 3$ corresponds to the $(3,0)$-regular set $V(P)$, eigenvalue $\lambda = 1$ corresponds to the $(2,1)$-regular set $\{1,2,5,7,8\}$ (or to $\{3,4,6,9,10\}$) and eigenvalue $\lambda = -2$ corresponds to the $(1,3)$-regular set $\{5,6,7,8,9,10\}$ (or to the $(0,2)$-regular set $\{1,2,3,4\}$). Note that the Petersen graph is a strongly regular graph with parameters $(10,3,0,1)$.

Another strongly regular graph for which each of its distinct eigenvalues corresponds to a $(k,\tau)$-regular set is the Hoffman-Singleton graph $HS$ (a strongly regular graph with parameters $(50,7,0,1)$): its index $\lambda = 7$ corresponds to the $(7,0)$-regular set $V(HS)$, the second eigenvalue $\lambda = 2$ corresponds to one of the five $(3,1)$-regular sets in which the vertex set $V(HS)$ can be partitioned (each one inducing a Petersen graph), and the third eigenvalue $\lambda = -3$ corresponds to a maximum independent set which is $(0,3)$-regular (see Lemma 5.9.1 in Godsil and Royle [12]).

Not all strongly regular graphs have this property. Further, there are also regular graphs that are not strongly regular that do have this property. For instance, such are regular graphs with four distinct eigenvalues: the 3-dimensional cube $Q_3$ has 4 distinct eigenvalues $-3, -1, 1$ and 3 corresponding, respectively, to a $(0,3), (1,2), (2,1)$ and $(3,0)$-regular set.

**Problem.** Find graphs in which each distinct adjacency eigenvalue corre-
sponds to a \((k, \tau)\)-regular set.

Note also that these graphs are particular cases of integral graphs.

**Problem AWGS.13** Graphs with stable least eigenvalue

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In this problem we are interested in graphs \(G\) satisfying

\[
(\forall v \in V(G)) \quad \lambda_{\min}(G - N_G(v)) = \lambda_{\min}(G). \tag{1}
\]

An example of such graph is shown in Figure 2. Note that from (1) it follows that also

\[
(\forall v \in V(G)) \quad \lambda_{\min}(G - \{v\}) = \lambda_{\min}(G).
\]

![Figure 2: A graph satisfying condition (1).](image)

**Problem.** Characterize graphs for which (1) holds.

The motivation for this problem lies in the recognition of graphs \(G\) in which the stability number is equal to the optimal value of a convex quadratic problem

\[
\max\{2\hat{e}^T x - x^T(\frac{A_G}{-\lambda_{\min}(A_G)} + I)x : x \geq 0\},
\]

where \(\hat{e}\) is the all ones vector and \(I\) denotes the identity matrix (see [7]).
References


