Resolution of AutoGraphiX conjectures relating the index and matching number of graphs

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Abstract

We resolve conjectures of AutoGraphiX relating the index and the matching number of connected graphs.

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1. Introduction

Let \( G = (V, E) \) be a simple graph with \( n \) vertices and \( m \) edges. Its adjacency matrix \( A \), indexed by the vertices of \( G \), is defined by \( A_{ij} = 1 \) if \( ij \in E \), and \( A_{ij} = 0 \) otherwise. The largest eigenvalue of \( A \) is the index of \( G \), denoted by \( \lambda_1 = \lambda_1(G) \). A matching in a graph is a set of disjoint edges, and the maximum cardinality of a matching over all possible matchings in a graph \( G \) is the matching number of \( G \), denoted by \( \mu = \mu(G) \).

AutoGraphiX [1, 2] is an optimization software aimed at finding extremal graphs satisfying given constraints. It can be used to find conjectures on relations between various graph parameters, and this approach to using AutoGraphiX has been elaborated in [3]. Here we are interested in the following conjecture appearing in [3], which is actually made up of three independent conjectures:

**Conjecture 7.** Let \( G \) be a connected graph on \( n \geq 3 \) vertices with index \( \lambda_1 \) and matching number \( \mu \). Then

\[
\lambda_1 - \mu \leq n - 1 - \lfloor n/2 \rfloor, \tag{Part A}
\]

with equality if and only if \( G \) is the complete graph \( K_n \). Also,

\[
\lambda_1 + \mu \geq \sqrt{n-1} + 1 \tag{Part B}
\]

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and
\[ \frac{\lambda_1}{\mu} \leq \sqrt{n-1}, \]
with equalities if and only if \( G \) is the star \( S_n \).

In the next three sections we resolve each part of the conjecture: part A turns out to be correct, part C is correct for all graphs \( G \neq K_3 \), while for part B there exist an infinite family of counterexamples. We also give a simple lower bound on the index of graphs with a given matching number \( \mu \).

2. Part A

Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be two graphs. The union \( G_1 \cup G_2 \) is the graph \( (V_1 \cup V_2, E_1 \cup E_2) \). The join \( G_1 \vee G_2 \) is obtained from \( G_1 \cup G_2 \) by joining with an edge each vertex of \( G_1 \) to each vertex of \( G_2 \).

Part A of Conjecture 7 is implied by a result on the maximum index of graphs with a given matching number \( \mu \).

**Theorem 1** ([4]). Let \( \mathcal{G}_{n,\mu} \) be the set of graphs on \( n \) vertices with matching number \( \mu \). For any \( G \in \mathcal{G}_{n,\mu} \), we have

1. If \( n = 2\mu \) or \( n = 2\mu + 1 \), then \( \lambda_1(G) \leq n - 1 \) with equality if and only if \( G = K_n \).
2. If \( 2\mu + 2 \leq n < 3\mu + 2 \), then \( \lambda_1(G) \leq 2\mu \) with equality if and only if \( G = K_{2\mu+1} \cup K_{n-2\mu-1} \).
3. If \( n = 3\mu + 2 \), then \( \lambda_1(G) \leq 2\mu \) with equality if and only if \( G = K_{2\mu+1} \cup K_{n-2\mu-1} \).
4. If \( 3\mu + 3 \leq n \), then \( \lambda_1(G) \leq \frac{1}{2}(\mu - 1 + \sqrt{(\mu - 1)^2 + 4\mu(n - \mu)}) \), with equality if and only if \( G = K_{\mu} \vee K_{n-\mu} \).

The cases \( 2\mu \leq n \leq 3\mu + 2 \) are straightforward to check, and the case \( 3\mu + 3 \leq n \) follows easily as well: from \( 4\mu(n - \mu) \leq n^2 \), we get that
\[ \sqrt{(\mu - 1)^2 + 4\mu(n - \mu)} \leq \sqrt{(\mu - 1)^2 + n^2} < \mu - 1 + n, \]
and so
\[ \lambda_1 - \mu < \frac{1}{2}(\mu - 1 + \mu - 1 + n) - \mu = \frac{n}{2} - 1 \leq n - 1 - \lfloor n/2 \rfloor. \]
3. Part C

From Theorem 1 we see that in cases $2\mu \leq n \leq 3\mu + 2$ it always holds that $\frac{\lambda_1}{\mu} \leq 2$, which is smaller than or equal to $\sqrt{n-1}$, unless $n \leq 4$. A short look at the table of connected graphs with three and four vertices yields a unique counterexample to part C of Conjecture 7: a triangle $K_3$ with $\lambda_1 = 2$ and $\mu = 1$. It is hard to believe that AutoGraphiX has skipped a triangle in its search for extremal graphs. More probable, the authors of Conjecture 7 instructed AutoGraphiX to search for graphs with larger number of vertices (five or more), and have thus skipped such an obvious counterexample.

The case $3\mu + 3 \leq n$ follows easily from Theorem 1 as well. For $\mu = 1$, the graph is a star $S_n$ for which $\lambda_1 = \sqrt{n-1}$, while for $2 \leq \mu$ we have

$$\frac{\lambda_1}{\mu} \leq \frac{\mu - 1}{2\mu} + \sqrt{\frac{n}{\mu} - 1 + \left(\frac{\mu - 1}{2\mu}\right)^2} < \frac{1}{2} + \sqrt{\frac{n-3}{4}} \leq \sqrt{n-1}.$$

4. Part B

For this part of Conjecture 7 we first construct an infinite family of counterexamples, and then prove a modified version of the conjecture with a correct order of magnitude.

Let $n$ and $\mu$ be chosen freely, such that $n$ is sufficiently larger than $\mu$. Let $S_1, S_2, \ldots, S_\mu$ be the stars, each with either $\lfloor \frac{n-1}{\mu} \rfloor + 1$ or $\lceil \frac{n-1}{\mu} \rceil + 1$ vertices, so that their total number of vertices is $n + \mu - 1$. Further, let $T$ be an arbitrary tree on vertex set $\{1, 2, \ldots, \mu\}$, with its maximum degree being at most $\lfloor \frac{n-1}{\mu} \rfloor$. The new tree $T^*$ is formed from $T$ first by replacing vertex $i$ of $T$ with star $S_i$, for each $1 \leq i \leq \mu$, and then by identifying a distinct pair of leaves, one from $S_i$ and one from $S_j$, for each edge $ij$ in $T$. An example of tree $T^*$ obtained for $n = 17, \mu = 4$ and $T$ being a star on four vertices, is shown in Fig. 1.

![Figure 1: An example of tree $T^*$. Darker vertices represent pairs of leaves that were identified in forming $T^*$.

The tree $T^*$ obtained in this way has $n$ vertices, the matching number $\mu(T^*) = \mu$, since the edge set of $T^*$ is partitioned into $\mu$ stars, and the maximum vertex degree $\Delta(T^*) = \lceil \frac{n-1}{\mu} \rceil$. It was shown in [6] that for trees holds...
\( \lambda_1(T^*) \leq 2\sqrt{\Delta(T^*)} \), and thus

\[
\lambda_1(T^*) + \mu(T^*) \leq 2 \sqrt{\left\lceil \frac{n-1}{\mu} \right\rceil} + \mu.
\]

If we suppose that \( \mu = \Theta(n^{1/3}) \) (or, in particular, take \( \mu = \lfloor \sqrt[3]{n-1} \rfloor \)), we obtain that \( \lambda_1(T^*) + \mu(T^*) \) is \( O(n^{1/3}) \), and so it will be strictly smaller than \( \sqrt{n-1} + 1 \) for each sufficiently large value of \( n \), yielding to infinitely many counterexamples.

In particular, for \( n = 600, \mu = 8 \) and \( T \) being a path on eight vertices, we get that

\[
\lambda_1(T^*) + \mu(T^*) \leq 2 \sqrt{\left\lceil \frac{599}{8} \right\rceil} + 8 < 25.3206 < 25.4744 < \sqrt{599} + 1.
\]

Nevertheless, we are able to prove the following lower bound on the index of graphs with a given matching number.

**Theorem 2.** Let \( G \) be a graph with \( m \) edges and matching number \( \mu \). Then

\[
\lambda_1(G) \geq \sqrt{\left\lceil \frac{m+\mu}{2\mu} \right\rceil}.
\]

**Proof.** Let \( M \) be a matching of \( G \) with \( \mu \) edges. Since \( M \) cannot be extended to a larger matching, each edge \( e \notin M \) is incident to one or two edges in \( M \). Thus, the sum of degrees of \( 2\mu \) vertices belonging to edges in \( M \) is at least \( m + \mu = 2\mu + (m - \mu) \), where \( 2\mu \) comes from edges in \( M \) and \( m - \mu \) is the contribution of edges not in \( M \). Therefore, at least one of these \( 2\mu \) vertices has degree at least \( \left\lceil \frac{m+\mu}{2\mu} \right\rceil \). Then from a well-known result \( \lambda_1(G) \geq \sqrt{\Delta(G)} \) [5], where \( \Delta(G) \) is the maximum vertex degree of \( G \), we have

\[
\lambda_1(G) \geq \sqrt{\Delta(G)} \geq \sqrt{\left\lceil \frac{m+\mu}{2\mu} \right\rceil}.
\]

From Theorem 2 and the inequality between arithmetic and geometric means, we have the following lower bound on \( \lambda_1(G) + \mu(G) \):

\[
\lambda_1 + \mu \geq \sqrt{\left\lceil \frac{m+\mu}{2\mu} \right\rceil} + \mu \geq \frac{1}{2} \sqrt{\frac{m+\mu}{2\mu}} + \frac{1}{2} \sqrt{\frac{m+\mu}{2\mu}} + \mu
\]

\[
\geq \frac{3}{2} \sqrt{\frac{m+\mu}{2\mu}} = \frac{3}{2} \sqrt{m+\mu}.
\]

From this corollary, we conclude that part B of Conjecture 7 nevertheless holds for all graphs with \( m + \mu \geq \frac{1}{27} ((n + 2)\sqrt{n-1} + 3n - 2) \).
5. Concluding remarks

A good deal of results in extremal graph theory concentrates on finding graphs with extremal value of one invariant under the constraint that the other (integer) invariant is kept constant. It would be worthwhile to investigate to what extent AutoGraphiX can be used in obtaining such type of conjectures, when the integer invariant to be kept constant, is a nontrivial one: clique number, chromatic number, radius, diameter, matching number etc. Results in this direction already exist in the literature [7, 8, 9]. This is a more standard form of results than the so-called AGX Form 1 [3]. Although some strong and interesting results were found in this form (see the AutoGraphiX series of papers [10, 11, 12]), it is less intuitive and very often leads to easy or expected results.

References


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