LARGE SETS OF NONCOSPECTRAL GRAPHS WITH EQUAL LAPLACIAN ENERGY

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Abstract

Several alternative definitions to graph energy have appeared in literature recently, the first among them being the Laplacian energy, defined by Gutman and Zhou in [Linear Algebra Appl. 414 (2006), 29–37]. We show here that Laplacian energy apparently has small power of discrimination among threshold graphs, by showing that, for each \( n \), there exists a set of \( n \) mutually noncospectral connected threshold graphs with equal Laplacian energy with \( O(\sqrt{n}) \) vertices only. Nevertheless, situation becomes opposite when trees are considered, as it turns out that, up to 20 vertices, there exists no pair of noncospectral trees with equal Laplacian energies.

1 Introduction

Let \( G = (V, E) \) be a finite, simple, undirected graph with vertices \( V = \{1, 2, \ldots, n\} \) and \( m = |E| \) edges. The degree of a vertex \( u \in V \) will be denoted by \( d_u \). Let \( G \) have adjacency matrix \( A \) with eigenvalues \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \), and Laplacian matrix \( L = D - A \), where \( D \) is the diagonal matrix of vertex degrees, with eigenvalues \( \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n = 0 \).

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Additional details on the theory of graph spectra may be found in [1]. These eigenvalues obey the following well-known relations:

\[ \sum_{i=1}^{n} \lambda_i = 0, \quad \sum_{i=1}^{n} \mu_i = 2m. \] (1)

The energy and the Laplacian energy of \( G \) are now defined as follows

\[ E = E(G) = \sum_{i=1}^{n} |\lambda_i|, \quad LE = LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|. \] (2)

From (1) and (2) we can observe that both energies represent the absolute deviation of corresponding eigenvalues from their average value. Thus, we can introduce the following

**Definition 1** The energy of a given matrix \( M \), denoted as \( M \)-energy, is the absolute deviation of eigenvalues of \( M \) from their average value.

This way, the energy of a graph is its \( A \)-energy and the Laplacian energy of a graph is its \( L \)-energy. Other types of energy can be defined in the same way, the difference being only in the matrix under consideration: for example, the energy of a distance matrix is studied in [11, 12]. Among those found in literature, it is the Laplacian-like energy only, defined by Liu and Liu [13], that does not fit this setting (which, at the end, may happen to be to its advantage, as a number of extremal problems for Laplacian-like energy can be solved by considering the coefficients of characteristic polynomial of \( L \) and finding transformations which are monotone on these coefficients [14, 15, 16]).

On the other hand, Nikiforov [17] has recently introduced another concept of the energy of a complex matrix \( M \) as the sum of the singular values of \( M \), which made possible to determine the energy of random graphs.

A feasible use of energies, as numerical invariants, is to distinguish nonisomorphic graphs from each other. In that respect, for a given type of graph matrix \( M \), graphs having equal \( M \)-energy will be called \( M \)-equienergetic. Of course, since \( M \)-energy is calculated from spectrum of \( M \), \( M \)-cospectral graphs will trivially have the same \( M \)-energy. Thus, for a given type of graph matrix \( M \), two graphs \( G \) and \( H \) will be called \( M \)-equienergetic if they are not \( M \)-cospectral, yet have equal \( M \)-energies.

**Definition 2** For a given type of graph matrix \( M \), two graphs \( G \) and \( H \) will be called \( M \)-equienergetic if they are not \( M \)-cospectral, yet have equal \( M \)-energies.

A number of results on \( A \)-equienergetic graphs have appeared recently [18]-[27]. In principle, most of these results show that \( A \)-equienergetic graphs exist in various classes of graphs, and in some cases, sets of \( n \) such graphs can be found (e.g., see [21]), although on very large number of vertices (of order \( 5^n \) in [21]).

When it comes to Laplacian energy, it appears that it might not be well suited to distinguish among nonisomorphic graphs, as there exists a triplet of \( L \)-equienergetic graphs on four vertices already:
Our main task here is to show that the above example is not a coincidence. In particular, we show that for any \( n \in \mathbb{N} \) there exists a set of \( n \) \( L \)-equienergetic threshold graphs on \( O(\sqrt{n}) \) vertices only. It turns out that these graphs have equal number of edges as well. We find them in the class of threshold graphs.

## 2 Threshold graphs

Threshold graphs are a simple class of graphs, which due to their wide applicability, keeps reappearing under various names. A good survey on the properties of threshold graphs is \([28]\).

Basically, a threshold graph is obtained in a recursive process, where one starts with an isolated vertex and at each step either a new isolated vertex is added or a new vertex adjacent to all previous vertices is added. This construction process can be encoded with a sequence of 0s and 1s, where 0 represents addition of an isolated vertex, while 1 represents addition of a vertex adjacent to all previous vertices. Thus, an \( n \)-vertex threshold graph can be encoded with a sequence of \( n - 1 \) symbols. For our purposes, we will extend this encoding with an arbitrary initial element (0 or 1) that will correspond to the starting isolated vertex. Thus, in our case, an \( n \)-vertex threshold graph will be encoded with a sequence of \( n \) symbols, where symbol at position \( k \) describes the nature of vertex \( k \). It is immediate to see from this encoding that two threshold graphs are isomorphic if and only if they have the same encoding sequence (without initial element).

Suppose that \( G \) is a threshold graph with encoding sequence \( a_1a_2\ldots a_n \in \{0, 1\}^n \), and let \( d_1 \geq d_2 \geq \ldots \geq d_n \) be its degree sequence. The degree sequence can be represented via its Ferrers diagram, as shown by example in Fig. 1.

A particularly nice property of a class of threshold graphs is that

**Lemma 1 ([28])** The Laplacian spectrum of a threshold graph \( G \) is the conjugate of its degree sequence.

Thus, Laplacian spectrum of \( G \) consists of eigenvalues

\[
\mu_i = |\{j : d_j \geq i\}|, \quad i = 1, 2, \ldots, n. \tag{3}
\]

Let us now study the effects of two particular operations on the encoding sequence of a threshold graph:

**Operation A.** Changing 01 with 10 in encoding sequence.

Suppose \( a_i = 0 \) and \( a_{i+1} = 1 \) and form a new threshold graph \( G' \) encoded by a sequence \( a_1 \ldots a_{i-1}10a_{i+2} \ldots a_n \). Then \( G' \) is obtained by removing edge \( \{i, i + 1\} \) from \( G \), so that the degrees of \( i \) and \( i + 1 \) decrease by one, while the degrees of all other vertices remain
intact. Then from (3) we see that, in the Laplacian spectrum of $G'$, the eigenvalues $\mu_d$ and $\mu_d'$ decrease by one, while other eigenvalues remain intact.

**Operation B. Changing 10 with 01 in encoding sequence.**

Suppose $a_i = 1$ and $a_{i+1} = 0$ and form a new threshold graph $G''$ encoded by a sequence $a_1 \ldots a_{i-1}01a_{i+2} \ldots a_n$. Then $G''$ is obtained by adding edge $\{i, i+1\}$ to $G$, so that the degrees of $i$ and $i+1$ increase by one, while the degrees of all other vertices remain intact. Then from (3) we see that, in the Laplacian spectrum of $G''$, the eigenvalues $\mu_d'_{i+1}$ and $\mu_d''_{i+1}$ increase by one, while other eigenvalues remain intact.

## 3 A particular threshold graph and its mates

If we apply just one of operations $A$ and $B$ to a threshold graph $G$, then its number of edges changes, and so, the term $2m/n$ in the formula for $L$-energy changes as well. In such case, if we wish to know how does $L$-energy change, we first need to know how many eigenvalues of $L$ fall into each of intervals $[0, 2(m-1)/n], [2(m-1)/n, 2m/n]$ and $[2m/n, +\infty]$. However, we can work around this problem if we simultaneously apply both operations to $G$, as then the number of edges remain the same.

The threshold graph $G_k$, $k \geq 3$, that will be used to generate a large set of $L$-equienergetic graphs, has encoding sequence

$$010101 \ldots 010101.$$ \hspace{1cm} (4)

For example, $G_5$ is the first graph shown in Fig. 2.

From its encoding sequence, it is easy to see that $G_k$ has $2k$ vertices with degrees, given in the same order as vertices,

$$k, k, k-1, k+1, k-2, k+2, \ldots, 1, 2k-1,$$

where the vertex at position $2i-1$, corresponding to 0 in (4), has degree $k-i+1$, while the vertex at position $2i$, corresponding to 1 in (4), has degree $k+i-1$, $i = 1, 2, \ldots, k$. 

Figure 1: An example of a threshold graph.
From (3), its Laplacian spectrum is
\[ [2k, 2k - 1, \ldots, k + 2, k + 1, k - 1, k - 2, \ldots, 1, 0]. \]
The average degree of \( G_k \) is exactly \( k \), and thus, its Laplacian energy is equal to \( k(k + 1) \).

Let us now apply operation \( A \) to pair 01 at positions \( 2i - 1 \) and \( 2i \), \( 1 \leq i \leq k \), and operation \( B \) to pair 10 at positions \( 2j \) and \( 2j + 1 \), \( 1 \leq j \leq k - 1 \), in the encoding sequence (4) (assuming that \( \{2i - 1, 2i\} \cap \{2j, 2j + 1\} = \emptyset \)). Then the degrees of vertices \( 2i - 1 \) and \( 2i \) decrease by one, while those of vertices \( 2j \) and \( 2j + 1 \) increase by one, so that the new degree sequence becomes
\[
\begin{align*}
\text{vertices 1 to 2i - 2:} & \quad k, k, \ldots, k - i + 2, k + i - 2, k - i, k + i, \ldots, k - j + 1, k + j, k - j + 1, k + j, \ldots, 1, 2k - 1, \\
\text{vertices 2i + 1 to 2j - 1:} & \quad k - i, k + i - 2, k - i, k + i, \ldots, k - j + 1, k + j, k - j + 1, k + j, \ldots, 1, 2k - 1, \\
\text{vertices 2j + 2 to 2k:} & \quad k + j, k - j + 1, k + j, \ldots, 1, 2k - 1.
\end{align*}
\]

These operations decrease or increase by one corresponding Laplacian eigenvalues of \( G_k \), so that their new values are
\[
\begin{align*}
\mu'_{k-i+1} &= k + i - 1, \quad (5) \\
\mu'_{k-j+1} &= k + j + 1, \quad (6) \\
\mu'_{k+i-1} &= k - i, \quad (7) \\
\mu'_{k+j} &= k - j + 1, \quad (8)
\end{align*}
\]
while the rest of the \( L \)-spectrum remains intact. Thus, two \( L \)-eigenvalues that were larger than \( k \) increase and decrease by one, respectively, and two \( L \)-eigenvalues that were smaller than \( k \) increase and decrease by one, respectively. Since the average degree remains \( k \) in a newly obtained graph, we see that the \( L \)-energy does not change, and so, \( G_k \) and a newly obtained graph are \( L \)-equienergetic.

Next, notice that operation \( A \) can be applied to \( G_k \) in \( k - 1 \) ways (we do not apply it to last pair 01 in the encoding sequence, as it results in a disconnected graph). If \( A \) is applied to the first pair 01, then operation \( B \) can be applied in \( k - 2 \) ways, while in other cases, \( B \) can be applied in \( k - 3 \) ways. Thus, we can apply them simultaneously in \( k^2 - 4k + 4 \) ways. Since all these threshold graphs have distinct encoding sequences, no two of them are isomorphic. Moreover, it is easy to see from (5)-(8) that no two of them may be \( L \)-cospectral as well. Thus, we have just shown

**Theorem 1** For each \( k \geq 3 \), there exists a set of \( k^2 - 4k + 5 \) \( L \)-equienergetic graphs on \( 2k \) vertices.

The construction of such set is exemplified for \( k = 5 \) in Fig. 2.

### 4 Concluding remarks

We see from Theorem 1 that there exist arbitrarily large sets of \( L \)-equienergetic graphs, with a relatively small number of vertices, and moreover with the same number of edges, all of which show that \( L \)-energy is not well suited to distinguish threshold graphs from each other. This conclusion may extend to connected graphs as well, as it turns out that there exists
Figure 2: $k^2 - 4k + 5 = 10$ $L$-equienergetic graphs on $2k = 10$ vertices, with encoding sequence and Laplacian spectrum below each graph.

- 297 pairs of noncospectral $L$-equienergetic graphs among 853 connected graphs on seven vertices,
- 13044 pairs among 11117 connected graphs on eight vertices (implying, in fact, existence of a large number of $L$-equienergetic sets containing at least four graphs each), and
- 39304 pairs among 261080 connected graphs on nine vertices.
Nevertheless, $L$-energy may be fit for business in more restricted graph classes. For example, our computer search employing first a Java-based program that puts trees in hash map with $L$-energy as keys, calculated using Colt library (available from http://acs.lbl.gov/~hoschek/colt/) and then checking the findings with Wolfram’s Mathematica, revealed that

there is not a single pair of $L$-equienergetic trees up to 20 vertices!

There is a large number of trees whose $L$-energy differs by less than $10^{-11}$, but no two of them are really $L$-equienergetic. This is in sharp contrast with the fact that there are already 120 pairs of equienergetic trees on 20 vertices (which was checked by the same programs, except that $A$-energy was calculated instead of $L$-energy).

Of course, it is hard to believe that there exists no pair of $L$-equienergetic trees at all, and they probably do exist on a larger number of vertices. Thus, we close this paper by leaving the following

Open problem. Find a pair of $L$-equienergetic trees.

References


