On the Wiener index and Laplacian coefficients of graphs with given diameter or radius *

Aleksandar Ilić ‡
Faculty of Sciences and Mathematics, University of Niš, Serbia
e-mail: aleksandari@gmail.com

Andreja Ilić
Faculty of Sciences and Mathematics, University of Niš, Serbia
e-mail: ilic_andrejko@yahoo.com

Dragan Stevanović
University of Primorska—FAMNIT, Glagoljaška 8, 6000 Koper, Slovenia
Mathematical Institute, Serbian Academy of Science and Arts,
Knez Mihajlova 36, 11000 Belgrade, Serbia
e-mail: dragance106@yahoo.com

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Abstract

Let $G$ be a simple undirected $n$-vertex graph with the characteristic polynomial of its Laplacian matrix $L(G)$, $\det(\lambda I - L(G)) = \sum_{k=0}^{n} (-1)^k c_k \lambda^{n-k}$. It is well known that for trees the Laplacian coefficient $c_{n-2}$ is equal to the Wiener index of $G$. Using a result of Zhou and Gutman on the relation between the Laplacian coefficients and the matching numbers in subdivided bipartite graphs, we characterize first the trees with given diameter and then the connected graphs with given radius which simultaneously minimize all Laplacian coefficients. This approach generalizes recent results of Liu and Pan [MATCH Commun. Math. Comput. Chem. 60 (2008), 85–94] and Wang and Guo [MATCH Commun. Math. Comput. Chem. 60 (2008), 609–622] who characterized $n$-vertex trees with fixed diameter $d$ which minimize the Wiener index. In conclusion, we illustrate on examples with Wiener and modified hyper-Wiener index that the opposite problem of simultaneously maximizing all Laplacian coefficients has no solution.

1 Introduction

Let $G = (V, E)$ be a simple undirected graph with $n = |V|$ vertices. The Laplacian polynomial $P(G, \lambda)$ of $G$ is the characteristic polynomial of its Laplacian matrix $L(G) = D(G) - A(G)$,

$$P(G, \lambda) = \det(\lambda I_n - L(G)) = \sum_{k=0}^{n} (-1)^k c_k \lambda^{n-k}.$$

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‡Corresponding author. If possible, send your correspondence via e-mail. Otherwise, snail-mail address is: Department of Mathematics and Informatics, Faculty of Sciences and Mathematics, Višegradska 33, 18000 Niš, Serbia
The Laplacian matrix $L(G)$ has non-negative eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \mu_n = 0$ [2]. From Viette’s formulas, $c_k = \sigma_k(\mu_1, \mu_2, \ldots, \mu_{n-1})$ is a symmetric polynomial of order $n-1$. In particular, $c_0 = 1$, $c_1 = 2n$, $c_n = 0$ and $c_{n-1} = n\tau(G)$, where $\tau(G)$ denotes the number of spanning trees of $G$. If $G$ is a tree, coefficient $c_{n-2}$ is equal to its Wiener index, which is a sum of distances between all pairs of vertices.

Let $m_k(G)$ be the number of matchings of $G$ containing exactly $k$ independent edges. The subdivision graph $S(G)$ of $G$ is obtained by inserting a new vertex of degree two on each edge of $G$. Zhou and Gutman [17] proved that for every acyclic graph $T$ with $n$ vertices

$$c_k(T) = m_k(S(T)),$$

$0 \leq k \leq n$. (1)

Let $C(a_1, \ldots, a_{d-1})$ be a caterpillar obtained from a path $P_d$ with vertices $\{v_0, v_1, \ldots, v_d\}$ by attaching $a_i$ pendant edges to vertex $v_i$, $i = 1, \ldots, d-1$. Clearly, $C(a_1, \ldots, a_{d-1})$ has diameter $d$ and $n = d + 1 + \sum_{i=1}^{d-1} a_i$. For simplicity, $C_{n,d} = C(0, \ldots, 0, a_{\lfloor d/2 \rfloor}, 0, \ldots, 0)$.

![Figure 1: Caterpillar $C_{n,d}$.](image)

In [12] it is shown that caterpillar $C_{n,d}$ has minimal spectral radius (the greatest eigenvalue of adjacency matrix) among graphs with fixed diameter.

Our goal here is to characterize the trees with given diameter and the connected graphs with given radius which simultaneously minimize all Laplacian coefficients. We generalize recent results of Liu and Pan [10], and Wang and Guo [15] who proved that the caterpillar $C_{n,d}$ is the unique tree with $n$ vertices and diameter $d$, that minimizes Wiener index. We also deal with connected $n$-vertex graphs with fixed diameter, and prove that $C_{n,2r-1}$ is extremal graph.

After a few preliminary results in Section 2, we prove in Section 3 that a caterpillar $C_{n,d}$ minimizes all Laplacian coefficients among $n$-vertex trees with diameter $d$. In particular, $C_{n,d}$ minimizes the Wiener index and the modified hyper-Wiener index among such trees. Further, in Section 4 we prove that $C_{n,2r-1}$ minimizes all Laplacian coefficients among connected $n$-vertex graphs with radius $r$. Finally, in conclusion we illustrate on examples with Wiener and modified hyper-Wiener index that the opposite problem of simultaneously maximizing all Laplacian coefficients has no solution.

## 2 Preliminaries

The distance $d(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest path between them. The eccentricity $\varepsilon(v)$ of a vertex $v$ is the maximum distance from $v$ to any other vertex.

**Definition 2.1** The diameter $d(G)$ of a graph $G$ is the maximum eccentricity over all vertices in a graph, and the radius $r(G)$ is the minimum eccentricity over all $v \in V(G)$.

Vertices of minimum eccentricity form the center (see [4]). A tree $T$ has exactly one or two adjacent center vertices. For a tree $T$,

$$d(T) = \begin{cases} 2r(T) - 1 & \text{if } T \text{ is bicentral}, \\ 2r(T) & \text{if } T \text{ has unique center vertex}. \end{cases}$$

(2)
The next lemma counts the number of matchings in a path $P_n$.

**Lemma 2.1** For $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, the number of matchings with $k$ edges for path $P_n$ is

$$m_k(P_n) = \binom{n-k}{k}.$$  

**Proof:** If $v$ is a pendant vertex of a graph $G$, adjacent to $u$, then for the matching number of $G$ the recurrence relation holds

$$m_k(G) = m_k(G - v) + m_{k-1}(G - u - v).$$

If $G$ is a path, then $m_k(P_n) = m_k(P_{n-1}) + m_{k-1}(P_{n-2})$. For base cases $k = 0$ and $k = 1$ we have $m_0(P_n) = 1$ and $m_1(P_n) = n - 1$. After substituting formula for $m_k(P_n)$, we get the well-known identity for binomial coefficients.

$$\binom{n-k}{k} = \binom{n-1-k}{k} + \binom{n-2-(k-1)}{k-1}.$$  

Maximum cardinality of a matching in the path $P_n$ is $\left\lfloor \frac{n}{2} \right\rfloor$ and thus, $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$. □

The union $G = G_1 \cup G_2$ of graphs $G_1$ and $G_2$ with disjoint vertex sets $V_1$ and $V_2$ and edge sets $E_1$ and $E_2$ is the graph $G = (V, E)$ with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. If $G$ is a union of two paths of lengths $a$ and $b$, then $G$ is disconnected and has $a + b$ vertices and $a + b - 2$ edges.

**Lemma 2.2** Let $m_k(a, b)$ be the number of $k$-matchings in $G = P_a \cup P_b$, where $a + b$ is fixed even number. Then, the following inequality holds

$$m_k\left(\left\lfloor \frac{a+b}{2} \right\rfloor, \left\lfloor \frac{a+b}{2} \right\rfloor\right) \leq \ldots \leq m_k(a + b - 2, 2) \leq m_k(a, b) = m_k(P_{a+b}).$$

**Proof:** Without loss of generality, we can assume that $a \geq b$. Notice that the number of vertices in every graph is equal to $a + b$. The path $P_{a+b}$ contains as a subgraph $P_{a'} \cup P_{b'}$, where $a' + b' = a + b$ and $a' \geq b' > 0$. This means that the number of $k$-matchings of $P_{a+b}$ is greater than or equal to the number of $k$ matchings of $P_{a'} \cup P_{b'}$, and therefore $m_k(a + b, 0) \geq m_k(a', b')$. In the sequel, we exclude $P_{a+b}$ from consideration.

For the case $k = 0$, by definition we have identity $m_0(G) = 1$. For $k = 1$ we have equality, because

$$m_1(a', b') = (a' - 1) + (b' - 1) = a' + b' - 2 = a + b - 2.$$  

We will use mathematical induction on the sum $a + b$. The base cases $a + b = 2, 4, 6$ are trivial for consideration using previous lemma. Suppose now that $a + b$ is an even number greater than 6 and consider graphs $G = P_a \cup P_b$ and $G' = P_{a'} \cup P_{b'}$, such that $a > b$ and $a' = a - 2$ and $b' = b + 2$. We divide the set of $k$-matchings of $G'$ in two disjoint subsets $M_1'$ and $M_2'$. The set $M_1'$ contains all $k$-matchings for which the last edge of $P_{a'}$ and the first edge of $P_{b'}$ are not together in the matching, while $M_2'$ consists of $k$-matchings that contain both the last edge of $P_{a'}$ and the first edge of $P_{b'}$. Analogously for the graph $G$, we define the partition $M_1 \cup M_2$ of the set of $k$-matchings.

Consider an arbitrary matching $M'$ from $M_1'$ with $k$ disjoint edges. We can construct corresponding matching $M$ in the graph $G$ in the following way: join paths $P_{a'}$ and $P_{b'}$ and form a path $P_{a+b-1}$ by identifying the last vertex on path $P_{a'}$ and the first vertex on path $P_{b'}$. This way we get a $k$-matching in path $P_{a+b-1} = v_1v_2\ldots v_{a+b-1}$. Next, split graph $P_{a+b-1}$ in two parts to get $P_a = v_1v_2\ldots v_a$ and $P_b = v_a'v_{a+1}\ldots v_{a+b-1}$. Note that the last edge in $P_a$ and the first edge in $P_b$ are not both in the matching $M$. This way we establish a bijection between sets $M_1$ and $M_1'$.  

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Now consider a matching $M'$ of $G'$ such that the last edge of $P_a'$ and the first edge of $P_b'$ are in $M'$. The cardinality of the set $M'_2$ equals to $m_{k-1}(a'-2,b'-2)$, because we cannot include the first two vertices from $P_a'$ and the last two vertices from $P_b'$ in the matching. Analogously, we conclude that $|M'_2| = m_{k-1}(a-2,b-2)$. This way we reduce problem to pairs $(a'-2,b'-2)$ and $(a-2,b-2)$ with smaller sum and inequality

$$m_{k-1}(a-2,b-2) \geq m_{k-1}(a'-2,b'-2)$$

holds by induction hypothesis. If one of numbers in the set \{a, b, a', b'\} becomes equal to 0 using above transformation, it must the smallest number $b$. In that case, we have $m_{k-1}(a-2,0) \geq m_{k-1}(a'-2,b'-2)$ which is already considered. □

**Lemma 2.3** For every $2 \leq r \leq \lfloor \frac{n}{2} \rfloor$, it holds

$$c_k(C_{n,2r}) \geq c_{k}(C_{n,2r-1}).$$

**Proof:** Coefficients $c_0$ and $c_n$ are constant, while trees $C_{n,2r}$ and $C_{n,2r-1}$ have equal number of vertices and thus, we have equalities

$$c_1(C_{n,2r}) = c_1(C_{n,2r-1}) = 2n \quad \text{and} \quad c_{n-1}(C_{n,2r}) = c_{n-1}(C_{n,2r-1}) = n.$$

Assume that $2 \leq k \leq n-2$. Using identity (1), we will establish injection from the set of $k$-matchings of subdivision graph $S(C_{n,2r-1})$ to $S(C_{n,2r})$. Let $v_0, v_1, \ldots, v_{2r-1}$ be the vertices on the main path of caterpillar $C_{n,2r-1}$ and $u_1, u_2, \ldots, u_{2r-2}$ pendent vertices from central vertex $v_r$. We obtain graph $C_{n,2r}$ by removing the edge $v_r u_{n-2r}$ and adding the edge $v_{2r-1} u_{n-2r}$. Assume that vertices $w_1, w_2, \ldots, w_{n-2r}$ are subdivision vertices of degree 2 on edges $v_r u_1, v_r u_2, \ldots, v_r u_{n-2r}$.

Consider an arbitrary matching $M$ of subdivision graph $S(C_{n,2r-1})$. If $M$ does not contain the edge $v_r w_{n-2r}$ then the corresponding set of edges in $S(C_{n,2r})$ is also a $k$-matching. Now assume that matching $M$ contains the edge $v_r w_{n-2r}$. If we exclude this edge from the graph $S(C_{n,2r-1})$, we get graph $G' = S(C_{n,2r-1}) - v_r w_{n-2r} = P_{2r} \cup P_{2r-2} \cup (n - 2r - 1)P_2 \cup P_1$. Therefore, the number of $k$-matchings that contain an edge $v_r w_{n-2r}$ in $S(C_{n,2r-1})$ is equal to the number of matchings with $k-1$ edges in graph $G'$ that is union of paths $P_{2r}$ and $P_{2(r-1)}$ and $n - 2r - 1$ disjoint edges $u_1 w_1, u_2 w_2, \ldots, u_{n-2r-1} w_{n-2r-1}$

$$S' = m_{k-1}(G') = m_{k-1}(P_{2r} \cup P_{2(r-1)} \cup (n - 2r - 1)P_2).$$

On the other side, let $G$ be the graph $S(C_{n,2r}) - v_{2r-1} w_{n-2r}$. Since $G$ contains as a subgraph $P_{2(r-1)} \cup (n - 2r - 1)P_2$, the number of $k$-matchings that contain the edge $v_{2r-1} w_{n-2r}$ is greater than or equal to the number of $(k-1)$-matchings in the union of path $P_{2(r-1)}$ and $n - 2r - 1$ disjoint edges. Therefore,

$$S = m_{k-1}(G) \geq m_{k-1}(P_{2(r-1)} \cup (n - 2r - 1)P_2).$$

Path $P_{2(r-1)}$ is obtained by adding an edge that connects the last vertex of $P_{2r}$ and the first vertex of $P_{2(r-1)}$, and thus we get inequality

$$m_k(S(C_{n,2r})) \geq m_k(S(C_{n,2r-1})).$$

Finally we get that all coefficients of Laplacian polynomial of $C_{n,2r}$ are greater than or equal to those of $C_{n,2r-1}$. □

The Laplacian coefficient $c_{n-2}$ of a tree $T$ is equal to the sum of all distances between unordered pairs of vertices, also known as the Wiener index,

$$c_{n-2}(T) = W(T) = \sum_{u,v \in V} d(u,v).$$
The Wiener index $s$ considered as one of the most used topological index with high correlation with many physical and chemical indices of molecular compounds. For recent surveys on Wiener index see [4], [5], [6]. The hyper-Wiener index $WW(G)$ [7] is one of the recently introduced distance based molecular descriptors. It was proved in [8] that a modification of the hyper-Wiener index, denoted as $WWW(G)$, has certain advantages over the original $WW(G)$. The modified hyper-Wiener index is equal to the coefficient $c_{n-3}$ of Laplacian characteristic polynomial.

**Proposition 2.4** The Wiener index of caterpillar $C_{n,d}$ equals:

$$W(C_{n,d}) = \begin{cases} 
\frac{d(d+1)(d+2)}{6} + (n - d - 1)(n - 1) + (n - d - 1) \left( \frac{d + 1}{2} \right) \frac{d}{2}, & \text{if } d \text{ is even,} \\
\frac{d(d+1)(d+2)}{6} + (n - d - 1)(n - 1) + (n - d - 1) \left( \frac{d + 1}{2} \right)^2, & \text{if } d \text{ is odd.}
\end{cases}$$

**Proof:** By summing all distances of vertices on the main path of length $d$, we get

$$\sum_{i=1}^{d} i(d + 1 - i) = (d + 1) \cdot \sum_{i=1}^{d} i - \sum_{i=1}^{d} i^2 = \frac{d(d+1)(d+2)}{6}.$$ 

For every pendant vertex attached to $v_{d/2} = v_c$ we have the same contribution in summation for the Wiener index:

$$(n - d - 2) + \left( \sum_{i=0}^{d} |i - c| + 1 \right) = (n - 1) + \sum_{i=0}^{d} |i - c|.$$ 

Therefore, based on parity of $d$ we easily get given formula. \qed 

### 3 Trees with fixed diameter

We need the following definition of $\sigma$-transformation, suggested by Mohar in [11].

**Definition 3.1** Let $u_0$ be a vertex of a tree $T$ of degree $p + 1$. Suppose that $u_0u_1, u_0u_2, \ldots, u_0u_p$ are pendant edges incident with $u_0$, and that $v_0$ is the neighbor of $u_0$ distinct from $u_1, u_2, \ldots, u_p$. Then we form a tree $T' = \sigma(T, u_0)$ by removing the edges $u_0u_1, u_0u_1, \ldots, u_0u_p$ from $T$ and adding $p$ new pendant edges $v_0v_1, v_0v_2, \ldots, v_0v_p$ incident with $v_0$. We say that $T'$ is a $\sigma$-transform of $T$.

Mohar proved that every tree can be transformed into a star by a sequence of $\sigma$-transformations.
Theorem 3.1 ([11]) Let $T' = \sigma(T, u_0)$ be a $\sigma$-transform of a tree $T$ of order $n$. For $d = 2, 3, \ldots k$, let $n_d$ be the number of vertices in $T - u_0$ that are at distance $d$ from $u_0$ in $T$. Then
\[
c_k(T) \geq c_k(T') + \sum_{d=2}^{k} n_d \cdot p \cdot \left( \frac{n - 2 - d}{k - d} \right) \quad \text{for } 2 \leq k \leq n - 2
\]
and $c_k(T) = c_k(T')$ for $k \in \{0, 1, n - 1, n\}$.

Theorem 3.2 Among connected acyclic graphs on $n$ vertices and diameter $d$, caterpillar
\[
C_{n,d} = C(0, \ldots, 0, a_{[d/2]}, 0, \ldots, 0),
\]
where $a_{[d/2]} = n - d - 1$, has minimal Laplacian coefficient $c_k$, for every $k = 0, 1, \ldots, n$.

Proof: Coefficients $c_0$, $c_1$, $c_{n-1}$ and $c_n$ are constant for all trees on $n$ vertices. The star graph $S_n$ is the unique tree with diameter 2 and path $P_n$ is unique graph with diameter $n - 1$. Therefore, we can assume that $2 \leq k \leq n - 2$ and $3 \leq d \leq n - 2$.

Let $P = v_0v_1v_2 \ldots v_d$ be a path in tree $T$ of maximal length. Every vertex $v_i$ on the path $P$ is a root of a tree $T_i$ with $a_i + 1$ vertices, that does not contain other vertices of $P$. We apply $\sigma$-transformation on trees $T_1, T_2, \ldots, T_{d-1}$ to decrease coefficients $c_k$, as long as we do not get a caterpillar $C(a_0, a_1, a_2, \ldots, a_d)$. By a theorem of Zhou and Gutman, it suffices to see that
\[
m_k(S(C(a_1, a_2, \ldots, a_{d-1}))) > m_k(S(C_{n,d})).
\]

Assume that $v_{[d/2]} = v_c$ is a central vertex of $C_{n,d}$. Let $u_1, u_2, \ldots, u_{n-d-1}$ be pendent vertices attached to $v_c$ in $S(C_{n,d})$, and let $w_1, w_2, \ldots, w_{n-d-1}$ be subdivision vertices on pendent edges $v_cu_1, v cw_2, \ldots, v cw_{n-d-1}$. We also introduce ordering of pendent vertices. Namely, in the graph $C_n(a_1, a_2, \ldots, a_{d-1})$ first $a_1$ vertices in the set $\{u_1, u_2, \ldots, u_{n-d-1}\}$ are attached to $v_1$, next $a_2$ vertices are attached to $v_2$, and so on.

Consider an arbitrary matching $M'$ with $k$ edges in caterpillar $S(C_{n,d})$. If $M$ does not contain any of the edges $\{v_iw_1, v_iw_2, \ldots, v_iw_{n-d-1}\}$, then we can a construct matching in $S(C(a_1, a_2, \ldots, a_{d-1}))$, by taking corresponding edges from $M$. If the edge $v_jw_i$ is in the matching $M'$ for some $1 \leq i \leq n - d - 1$, the corresponding edge $v_jw_i$ is attached to some vertex $v_j$, where $1 \leq j \leq d - 1$. Moreover, if we fix the number $l$ of matching edges in the set
\[
\{u_1w_1, u_2w_2, \ldots, u_{i-1}w_{i-1}, u_{i+1}w_{i+1}, \ldots, u_{n-1}w_{n-1}\},
\]
we have to choose exactly $k - l - 1$ independent edges in the remaining graphs. Caterpillar $S(C_{n,d})$ is decomposed into two path of lengths $2\left[\frac{d}{2}\right]$ and $2\left[\frac{d}{2}\right]$, and caterpillar $S(C(a_1, a_2, \ldots, a_{d-1}))$ is decomposed in paths of lengths $2j$ and $2d - 2j$. From Lemma 2.2 we can see that
\[
m_{k-l-1}(2\left[\frac{d}{2}\right], 2\left[\frac{d}{2}\right]) \leq m_{k-l-1}(2j, 2d - 2j).
\]

If we sum this inequality for $l = 0, 1, \ldots, k - 1$, we obtain that the number of $k$-matchings in graph $S(C_{n,d})$ is less than the number of $k$-matchings in $S(C(a_0, a_1, a_2, \ldots, a_d))$. Thus, for every tree $T$ on $n$ vertices with diameter $d$ holds:
\[
c_k(C_{n,d}) \leq c_k(T), \quad k = 0, 1, 2, \ldots, n.
\]

□
4 Graphs with fixed radius

Theorem 4.1 Among connected graphs on $n$ vertices and radius $r$, caterpillar $C_{n,2r-1}$ has minimal coefficient $c_k$, for every $k = 0, 1, \ldots, n$.

Proof: Let $v$ be a center vertex of $G$ and let $T$ be a spanning tree of $G$ with shortest paths from $v$ to all other vertices. Tree $T$ has radius $r$ and can be obtained by performing the breadth first search algorithm (see [3]). Laplacian eigenvalues of an edge-deleted graph $G - e$ interlace those of $G$, 

$$
\mu_1(G) \geq \mu_1(G - e) \geq \mu_2(G) \geq \mu_2(G - e) \geq \ldots \geq \mu_{n-1}(G) \geq \mu_{n-1}(G - e) \geq 0.
$$

Since, $c_k(G)$ is equal to $k$-th symmetric polynomial of eigenvalues $(\mu_1(G), \mu_2(G), \ldots, \mu_{n-1}(G))$, we have $c_k(G) \geq c_k(G - e)$. Thus, we delete edges of $G$ until we get a tree $T$ with radius $r$. This way we do not increase Laplacian coefficients $c_k$. The diameter of tree $T$ is either $2r - 1$ or $2r$. Since $c_k(C_{n,2r-1}) \leq c_k(C_{n,2r})$ from Lemma 2.3 we conclude that extremal graph on $n$ vertices, which has minimal coefficients $c_k$ for fixed radius $r$, is the caterpillar $C_{n,2r-1}$.

We can establish analogous result on the Wiener index.

Corollary 4.2 Among connected graphs on $n$ vertices and radius $r$, caterpillar $C_{n,2r-1}$ has minimal Wiener index.

5 Concluding remarks

We proved that $C_{n,2r-1}$ is the unique graph that minimize all Laplacian coefficients simultaneously among graphs on $n$ vertices with given radius $r$. In the class of $n$-vertex graphs with fixed diameter, we found the graph with minimal Laplacian coefficients in case of trees—because it is not always possible to find a spanning tree of a graph with the same diameter.

Naturally, one wants to describe $n$-vertex graphs with fixed radius or diameter with maximal Laplacian coefficients. We have checked all trees up to 20 vertices and classify them based on diameter and radius. For every triple $(n, d, k)$ and $(n, r, k)$ we found extremal graphs with $n$ vertices and fixed diameter $d$ or fixed radius $r$ that maximize coefficient $c_k$. The result is obvious—trees that maximize Wiener index are different from those with the same parameters that maximize modified hyper-Wiener index.

The following two graphs are extremal for $n = 18$ vertices with diameter $d = 4$; the first graph is a unique tree that maximizes Wiener index $c_{n-2} = 454$ and the second one is also a unique tree that maximizes modified hyper-Wiener index $c_{n-3} = 4960$. 

Figure 3: Correspondence between caterpillars $C_{n,d}$ and $C(a_0, a_1, \ldots, a_d)$. 

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Figure 4: Graphs with \( n = 18 \) and \( d = 4 \) that maximize \( c_{16} \) and \( c_{15} \).

The following two graphs are extremal for \( n = 17 \) vertices with radius \( r = 5 \); the first graph is a unique tree that maximizes Wiener index \( c_{n-2} = 664 \) and the second one is also a unique tree that maximizes modified hyper-Wiener index \( c_{n-3} = 9172 \).

Figure 5: Graphs with \( n = 17 \) and \( r = 5 \) that maximize \( c_{15} \) and \( c_{14} \).

References


