Remarks on \(Q\)-integral complete multipartite graphs\(^\star\)

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Abstract

A graph is \(Q\)-integral if the spectrum of its signless Laplacian matrix consists entirely of integers. In their study of \(Q\)-integral complete multipartite graphs, Zhao et al. [Linear Algebra Appl. 438 (2013), 1067–1077] posed two questions on the existence of such graphs. We resolve these questions and present some further results characterizing particular classes of \(Q\)-integral complete multipartite graphs.

Keywords: Signless Laplacian spectrum; Integral graphs; Complete multipartite graphs; Seidel spectrum.

2000 MSC: 05C50

1. Introduction

The study of graphs, whose spectrum of adjacency matrix consists of integers only, was initiated in a seminal paper by Harary and Schwenk in 1974 [1]. The results published up to 2002 have been surveyed in [2], while it should be noticed that more than a hundred new studies of integral graphs appeared in the last ten years, and that an updated survey of integral graphs is apparently due.

For a simple, undirected graph \(G\), the \textit{signless Laplacian} matrix of \(G\) is defined as \(Q(G) = D(G) + A(G)\), where \(D(G)\) is the diagonal matrix of the vertex degrees in \(G\) and \(A(G)\) is the adjacency matrix of \(G\). A graph \(G\) is called \(Q\)-integral if the characteristic polynomial of \(Q(G)\) has integer roots only. The \(Q\)-integral graphs were much less studied than integral graphs. Two simplest classes of \(Q\)-integral graphs are the complete graphs \(K_n\), with \(n \geq 2\), having the \(Q\)-spectrum \([2n - 2, (n - 2)^{n-1}]\), and the complete bipartite graphs \(K_{m,n}\), with \(m, n \geq 1\), having the \(Q\)-spectrum \([m + n, m^{n-1}, n^{m-1}, 0]\) (see [3]). By a computer search it is further established in [4] that there are exactly 172

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connected $Q$-integral graphs with up to ten vertices. In [5] it was proved that there are exactly 26 connected $Q$-integral graphs with maximum edge-degree at most four, with some partial results obtained for graphs with maximum degree five as well. The classes of $Q$-integral graphs obtained by the use of the join of regular graphs have been studied in [6] and, among others, all $Q$-integral complete split graphs have been identified there. A few infinite series of graphs having integer adjacency, Laplacian and signless Laplacian spectra have been constructed in [7], while semi-regular, bipartite $Q$-integral graphs have been considered in [8].

When it comes to the complete multipartite graphs, which in the case of integer adjacency spectrum have been the subject of earlier research [9, 10, 11], Yu et al [12] showed that the $Q$-characteristic polynomial of a complete multipartite graph $K_{p_1, \ldots, p_r}$ with $n = p_1 + \cdots + p_r$ vertices, is equal to

$$P(Q(K_{p_1, \ldots, p_r}), x) = \prod_{i=1}^{r} (x - n + p_i)^{p_i - 1} \prod_{i=1}^{r} (x - n + 2p_i) \left( 1 - \sum_{i=1}^{r} \frac{p_i}{x - n + 2p_i} \right). \quad (1)$$

If $p'_1, \ldots, p'_s$ denote all the distinct integers among $p_1, \ldots, p_r$ and $a_i, i = 1, \ldots, s$, denotes the multiplicity of $p'_i$ in the family $p_1, \ldots, p_r$, then $K_{p'_1, \ldots, p'_s}$ will also be denoted by $K_{a_1, p'_1, \ldots, a_s, p'_s}$. Zhao et al [13] have studied the question when $K_{a_1, p'_1, \ldots, a_s, p'_s}$ is $Q$-integral and, in particular, paid much attention to the cases $s = 2$ and $s = 3$. They have finished their study with two questions that we answer affirmatively here.

**Question 4.1** [13] Are there any $Q$-integral complete multipartite graphs $K_{a_1, p'_1, \ldots, a_s, p'_s}$ when $s \geq 4$?

**Question 4.2** [13] Are there any $Q$-integral complete multipartite graphs $K_{a_1, p'_1, \ldots, a_s, p'_s}$ with $a_1 = \cdots = a_s = 1$ when $s \geq 3$?

The paper is divided into sections according to the value of $s$ for studied complete multipartite graphs. In Section 2 we characterize a few particular classes of $Q$-integral complete multipartite graphs with $s = 2$ that were not considered by Zhao et al [13]. In Section 3 we give two infinite classes of $Q$-integral complete multipartite graphs with $s = 3$, one of which also satisfies $a_1 = a_2 = a_3 = 1$, hence affirmatively answering Question 4.2. In Section 4 we give another infinite class of $Q$-integral complete multipartite graphs with $s = 4$ and $a_1 = a_2 = a_3 = a_4 = 1$, as well as examples of $Q$-integral complete multipartite graphs with $s = 5$ and $s = 6$, hence affirmatively answering Question 4.1 as well.

2. **Characterization of some $Q$-integral complete multipartite graphs with $s = 2$**

From (1) it follows that

$$P(Q(K_{a,m}), x) = (x - m(a - 1))^{a(m-1)} (x - m(a - 2))^{a - 1} (x - 2m(a - 1)).$$
showing that \( K_{a,m} \) is \( Q \)-integral for every \( a, m \in \mathbb{N} \).

Recall that the join \( G_1 \sqcup G_2 \) of graphs \( G_1 \) and \( G_2 \) is obtained from the union of \( G_1 \) and \( G_2 \) by adding all possible edges joining a vertex in \( G_1 \) with a vertex in \( G_2 \). The following theorem from [6] gives a sufficient and necessary condition for the join of two \( Q \)-integral regular graphs to be \( Q \)-integral.

**Theorem 1 ([6]).** For \( Q \) integral of \( G \) showing that \( K \).

**Corollary 4.** \( K \) be \( \) from Corollary 2, the necessary and sufficient condition for \( K \).

**Corollary 3.** \( m,m,n \) is \( \) that, for some integer \( r \),

\[
(n_1 - 2r_1) - (n_2 - 2r_2))^2 + 4n_1n_2 \quad \text{is a perfect square.} \tag{2}
\]

Since \( K_{a,m,b,n} = K_{a,m} \sqcup K_{b,n} \). Theorem 1 applied to \( K_{a,m,b,n} \) yields

**Corollary 2.** \( K_{a,m,b,n} \) is \( Q \)-integral if and only if \( (m(a-2) - n(b-2))^2 + 4abmn \) is a perfect square.

The pairs of \( m \) and \( n \) satisfying the condition from the previous corollary can be easily characterized for small values of \( a \) and \( b \). In particular, for \( a = b = 2 \) we immediately have

**Corollary 3.** \( K_{m,m,n,n} \) is \( Q \)-integral if and only if \( mn \) is a perfect square.

**Corollary 4.** \( K_{m,m,n} \) is \( Q \)-integral if and only if there exist integers \( k \), \( p \) and \( q \) such that either \( (m,n) = (k(p^2 - q^2)/4, 2kq^2) \) or \( (m,n) = (kpq/2, k(p-q)^2) \).

**Proof.** From Corollary 2, the necessary and sufficient condition for \( K_{m,m,n} \) to be \( Q \)-integral is that, for some integer \( r \),

\[ n^2 + 8mn = r^2. \]

This is equivalent to \( (n+4m)^2 = r^2 + 16m^2 \), showing that \( r, 4m \) and \( n+4m \) form a Pythagorean triple. By Euclid’s formula (see, e.g., D.E. Joyce’s web version of Euclid’s Elements, Book X, Proposition 29 available at http://babbage.clarku.edu/~djoyce/java/elements/bookX/propX29.html), then there exist integers \( k \), \( p \) and \( q \) such that either

\[ r = 2kpq, \quad 4m = k(p^2 - q^2), \quad n + 4m = k(p^2 + q^2) \]

or

\[ r = k(p^2 - q^2), \quad 4m = 2kpq, \quad n + 4m = k(p^2 + q^2), \]

from where either \( (m,n) = (k(p^2 - q^2)/4, 2kq^2) \) or \( (m,n) = (kpq/2, k(p-q)^2) \).

**Corollary 5.** \( K_{m,m,m,n} \) is \( Q \)-integral if and only if there exist integers \( k, u \) and \( v \) such that \( m = k(u + v)(3u + v)/8 \) and \( n = k(u - v)(3u - v)/8 \).

**Proof.** Corollary 2 yields that the necessary and sufficient condition for \( K_{m,m,m,n} \) to be \( Q \)-integral is that, for some integer \( r \),

\[ (m + n)^2 + 12mn = r^2. \]
Adding $3(m-n)^2$ to both sides yields

$$(2m+2n)^2 = r^2 + 3(m-n)^2,$$  \hspace{1cm} (3)$$

which is an instance of a more general Diophantine equation

$$c^2 = a^2 + 3b^2.$$  \hspace{1cm} (4)$$

Let $k = \gcd(a,c)$. If $k > 1$, then $k^2|3b^2$ and, since 3 is square-free, $k|b$ as well, so that we may divide all terms in (4) by $k^2$ to get solutions $a' = a/k$ and $c' = c/k$ with $\gcd(a',c') = 1$. Suppose, therefore, that $\gcd(a,c) = 1$. From $3b^2 = (c-a)(c+a)$ and the fact that $\gcd(c-a,c+a) \in \{1,2\}$, depending on whether $c$ and $a$ are of the different or the same parity, the following four cases are possible:

Case (i) \hspace{0.5cm} $c-a = 3u^2$, $c+a = v^2$, $\gcd(u,v) = 1$, $c$ and $a$ are of the different parity. Then $a = (v^2 - 3u^2)/2$, $b = uv$ and $c = (3u^2 + v^2)/2$. Since $c$ and $a$ are of the different parity, both $u$ and $v$ have to be odd.

Case (ii) \hspace{0.5cm} $c-a = u^2$, $c+a = 3v^2$, $\gcd(u,v) = 1$, $c$ and $a$ are of the different parity. Then $a = (3v^2 - u^2)/2$, $b = uv$ and $c = (u^2 + 3v^2)/2$. Since $c$ and $a$ are of the different parity, both $u$ and $v$ have to be odd.

Case (iii) \hspace{0.5cm} $c-a = 6u^2$, $c+a = 2v^2$, $\gcd(u,v) = 1$, $c$ and $a$ are of the same parity. Then $a = v^2 - 3u^2$, $b = 2uv$ and $c = 3u^2 + v^2$.

Case (iv) \hspace{0.5cm} $c-a = 2u^2$, $c+a = 6v^2$, $\gcd(u,v) = 1$, $c$ and $a$ are of the same parity. Then $a = 3v^2 - u^2$, $b = 2uv$ and $c = u^2 + 3v^2$.

All solutions $(a,b,c)$ of (4) are then obtained by multiplying the solutions from Cases (i)-(iv) by an arbitrary integer $k$. Back to the original condition (3) with $b = m-n$ and $c = 2m+2n$, Cases (i) and (ii) yield

$$m = k(u+v)(3u+v)/8, \quad n = k(u-v)(3u-v)/8,$$

while Cases (iii) and (iv) yield

$$m = k(u+v)(3u+v)/4, \quad n = k(u-v)(3u-v)/4.$$

Since the first set of $(m,n)$ pairs properly contains the second set, we conclude that $K_{m,m,n,n}$ is $Q$-integral if and only if $m = k(u+v)(3u+v)/8$ and $n = k(u-v)(3u-v)/8$ for some integers $k,u,v$.

3. Infinite classes of $Q$-integral complete multipartite graphs with $s = 3$

We performed a computer search for $Q$-integral complete multipartite graphs with $s = 3$ in two classes of such graphs: $K_{p_1,p_2,p_3}$ and $K_{2p_1,p_2,p_3}$. There are quite a few $Q$-integral graphs among them: 312 $Q$-integral graphs $K_{p_1,p_2,p_3}$ with $1 \leq p_1 < p_2 < p_3 \leq 10000$, and 539 $Q$-integral graphs $K_{2p_1,p_2,p_3}$ with $1 \leq p_1 < p_2 < p_3 \leq 1000$. The parameters and $Q$-spectrum for several of these $Q$-integral graphs are given in Tables 1 and 2. It is easy to spot a pattern among the parameters of $Q$-integral $K_{2p_1,p_2,p_3}$ from Table 2.
by (1), has the form

\[ P = \text{characteristic polynomial of } G. \]

The conditions for \( k, s \in \mathbb{Z} \) are:

(i) \( k = 16k', r = k'(5 + 12s^2) \) for \( k', s' \in \mathbb{Z} \);

(ii) \( k = 48k', r = k'(15 + 4s^2) \) for \( k', s \in \mathbb{Z} \);

(iii) \( k = 3k' \) and \( r = k'(4s^2 \pm s + 1) \) for \( k', s \in \mathbb{Z} \);

(iv) \( r \equiv k(12t^2 \pm 9t + 2) \) for \( k, t \in \mathbb{Z} \);

(v) \( k = 6k', r = k'(8s^2 \pm 6s + 3) \) for \( k', s \in \mathbb{Z} \);

(vi) \( k = 2k', r = k'(24s'^2 \pm 6s' + 1) \) for \( k', s' \in \mathbb{Z} \).

If \( k \) is square-free, then \( K_{2,2k,r,3r-k} \) is \( Q \)-integral if and only if one of the above conditions holds.

**Proof.** Let \( n = 3k + 4r \). The \( Q \)-characteristic polynomial of \( K_{2,2k,r,3r-k} \), by (1), has the form

\[ P(Q(K_{2,2k,r,3r-k}), x) = (x - n + 2k)^{4k-2}(x - n + r)^{r-1}(x - n + 3r - k)^{3r-k-1}B(x), \]

Table 1: The parameters and \( Q \)-spectrum of \( Q \)-integral graphs \( K_{p_1,p_2,p_3} \) with \( p_1 \leq 20 \). The exponents in parentheses denote eigenvalue multiplicities.

<table>
<thead>
<tr>
<th>( p_1 )</th>
<th>( p_2 )</th>
<th>( p_3 )</th>
<th>( Q )-spectrum of ( K_{p_1,p_2,p_3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>72</td>
<td>847</td>
<td>([2, 79^{(6)}, 854^{(7)}, 882, 919^{(6)}, 968])</td>
</tr>
<tr>
<td>10</td>
<td>15</td>
<td>24</td>
<td>([9, 25^{(2)}, 34^{(14)}, 39^{(9)}, 64])</td>
</tr>
<tr>
<td>10</td>
<td>384</td>
<td>735</td>
<td>([9, 394^{(7)}, 745^{(3)}, 1024, 1119^{(9)}, 1255])</td>
</tr>
<tr>
<td>11</td>
<td>25</td>
<td>297</td>
<td>([3, 36^{(2)}, 308^{(24)}, 322^{(10)}, 363])</td>
</tr>
<tr>
<td>13</td>
<td>48</td>
<td>156</td>
<td>([9, 61^{(15)}, 169^{(4)}, 204^{(12)}, 256])</td>
</tr>
<tr>
<td>14</td>
<td>144</td>
<td>1694</td>
<td>([4, 158^{(1)}, 170^{(14)}, 1764, 1838^{(13)}, 1936])</td>
</tr>
<tr>
<td>20</td>
<td>30</td>
<td>48</td>
<td>([18, 50^{(4)}, 68^{(29)}, 78^{(19)}, 128])</td>
</tr>
<tr>
<td>20</td>
<td>768</td>
<td>1470</td>
<td>([18, 788^{(16)}, 1490^{(6)}, 2048, 2238^{(19)}, 2450])</td>
</tr>
</tbody>
</table>

Table 2: The parameters and \( Q \)-spectrum of \( Q \)-integral graphs \( K_{p_1,p_2,p_3} \) with \( p_1 \leq 5 \). The exponents in parentheses denote eigenvalue multiplicities.

<table>
<thead>
<tr>
<th>( p_1 )</th>
<th>( p_2 )</th>
<th>( p_3 )</th>
<th>( Q )-spectrum of ( K_{p_1,p_2,p_3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5</td>
<td>14</td>
<td>([3, 9^{(3)}, 16, 18^{(4)}, 19, 21^{(2)}, 31])</td>
</tr>
<tr>
<td>2</td>
<td>23</td>
<td>68</td>
<td>([3, 27^{(6)}, 72^{(22)}, 79, 91, 93^{(2)}, 112])</td>
</tr>
<tr>
<td>2</td>
<td>32</td>
<td>95</td>
<td>([3, 36^{(14)}, 96^{(3)}, 112, 127, 129^{(2)}, 151])</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>28</td>
<td>([6, 18^{(2)}, 32, 36^{(6)}, 38, 42^{(6)}, 62])</td>
</tr>
<tr>
<td>4</td>
<td>19</td>
<td>55</td>
<td>([6, 27^{(2)}, 62, 64^{(4)}, 74, 78^{(4)}, 104])</td>
</tr>
<tr>
<td>4</td>
<td>31</td>
<td>91</td>
<td>([6, 39^{(9)}, 96^{(3)}, 104, 122, 126^{(6)}, 158])</td>
</tr>
<tr>
<td>5</td>
<td>27</td>
<td>90</td>
<td>([7, 37^{(8)}, 100^{(27)}, 117, 122^{(6)}, 157])</td>
</tr>
</tbody>
</table>
where

\[
B(x) = (x - n + 4k)^2(x - n + 2r)(x - n + 2(3r - k)) \\
\left(1 - 2 \frac{2k}{x - n + 4k} - \frac{r}{x - n + 2r} - \frac{3r - k}{x - n + 2(3r - k)}\right).
\]

Hence, \(K_{2,2k,r,3r-k}\) is \(Q\)-integral if and only if \(B(x)\) has integer roots only. After replacing \(n = 3k + 4r\), \(B(x)\) simplifies to

\[
B(x) = (x - 3k)(x - 4r + k)(x^2 - (7k + 8r)x + 16(k^2 + kr + r^2)),
\]

showing that \(B(x)\) has integer roots if and only if the discriminant of a quadratic factor

\[
(7k + 8r)^2 - 64(k^2 + kr + r^2) = 48kr - 15k^2
\]
is a perfect square \(m^2\).

From \(48kr - 15k^2 = m^2\) we have \(k|5k^2 + m^2\). If \(k\) is square-free, then from \(k|m^2\) follows \(k|m\), i.e., \(m = km'\) holds.

Let us, therefore, suppose that \(m = km'\) for \(m' \in \mathbb{Z}\). If \(m' = 2s\) is even, then

\[
r = \frac{k(15 + 4s^2)}{48}.
\]

Since \(15 + 4s^2\) is odd, it may not be divisible by 16, so for \(r\) to be an integer, \(k\) has to be divisible by 16. If \(s = 3s'\), then \(r = k'(5 + 12s'^2)\) for \(k = 16k'\), yielding the condition (i). If \(s\) is not divisible by 3, then \(k\) has to be of the form \(k = 48k'\), so that \(r = k'(15 + 4s'^2)\), yielding the condition (ii).

If \(m'\) is odd, then it has the form \(m' = 8s + 1\) or \(m' = 8s + 3\) for \(s \in \mathbb{Z}\).

If \(m' = 8s + 1\), then

\[
r = k\frac{4s^2 + s + 1}{3},
\]

from where either \(3|k\) or \(s = 3t + 1\) for \(t \in \mathbb{Z}\). If \(k = 3k'\), then \(r = k'(4s^2 + s + 1)\), yielding the condition (iii). If \(s = 3t + 1\) for \(t \in \mathbb{Z}\), then \(r = k(12t^2 + 9t + 2)\), yielding the condition (iv).

If \(m' = 8s + 3\), then

\[
r = k\left(s^2 + 2s^2 + 3\right),
\]

from where either \(6|k\) or \(2|k, 3|s\). If \(k = 6k'\), then \(r = k'(8s^2 + 6s + 3)\), yielding the condition (v). If \(k = 2k'\) and \(s = 3s'\), then \(r = k'(24s'^2 + 6s' + 1)\), yielding the condition (vi).

\[\square\]

A bit more effort reveals a barely noticeable pattern among \(Q\)-integral complete tripartite graphs. Recall that the Fibonacci numbers are defined by the recurrence relation \(F_n = F_{n-1} + F_{n-2}, n \geq 2\), and the initial values \(F_0 = F_1 = 1\).
Theorem 7. The complete tripartite graph

\[ K_{F_{2n}^2 - F_{2n}, F_{2n}^3 + F_{2n}, F_{2n}^2 - 1} \]

is \( Q \)-integral for \( n \geq 2 \).

Proof. Let \( s = F_{2n} \) and \( p_1 = (s^2 - s)/2, p_2 = (s^2 + s)/2, p_3 = s^2 - 1 \) and \( n = p_1 + p_2 + p_3 = 2s^2 - 1 \). The characteristic polynomial of \( Q(K_{p_1,p_2,p_3}) \) is, by (1), equal to

\[
P(Q(K_{p_1,p_2,p_3}), x) = \prod_{i=1}^{3}(x-n+p_i)^{p_i-1} \prod_{i=1}^{3}(x-n+2p_i) \left( 1 - \sum_{i=1}^{3} \frac{p_i}{x-n+2pi} \right).
\]

The polynomial \( \prod_{i=1}^{3}(x-n+p_i)^{p_i-1} \) has integer roots only, so we focus further to

\[
\prod_{i=1}^{3}(x-n+2p_i) \left( 1 - \sum_{i=1}^{3} \frac{p_i}{x-n+2pi} \right) = x^3 - 2n x^2 + n^2 x - 4p_1p_2p_3.
\]

Substituting the values of \( p_1, p_2, p_3 \) and \( n \), we get

\[
x^3 - 2n x^2 + n^2 x - 4p_1p_2p_3 = (x-s^2) [x^2 - (3s^2 - 2)x + (s^2 - 1)^2],
\]

showing that \( s^2 = F_{2n}^2 \) is an integral root of \( P(Q(K_{p_1,p_2,p_3}), x) \). The roots of \( x^2 - (3s^2 - 2)x + (s^2 - 1)^2 \) are further equal to

\[
x_{1,2} = \frac{3s^2 - 2 \pm \sqrt{(3s^2 - 2)^2 - 4(s^2 - 1)^2}}{2} = \frac{3s^2 - 2 \pm s\sqrt{5s^2 - 4}}{2}.
\]

It is easy to show that \( 5F_{2n}^2 - 4 \) is a perfect square. First, we show that

\[
F_{2n}^2 - 1 = F_{2n+1}F_{2n-1}, \quad n \geq 1,
\]

by induction on \( n \). For \( n = 1 \), this reduces to \( F_2^2 - 1 = 3 = F_3F_1 \). Assuming that the equality \( F_{2n}^2 - 1 = F_{2n+1}F_{2n-1} \) holds for a particular \( n \geq 1 \), we have

\[
F_{2n+2}^2 - 1 = (F_{2n+1} + F_{2n})^2 - 1 = (F_{2n+1} + 2F_{2n})F_{2n+1} + F_{2n}^2 - 1 = (F_{2n+1} + 2F_{2n} + F_{2n-1})F_{2n+1} = (F_{2n+2} + F_{2n+1})F_{2n+1} = F_{2n+3}F_{2n+1}.
\]

If we now multiply (5) by four and add with \( F_{2n}^2 = (F_{2n+1} + F_{2n-1})^2 \), we have

\[
5F_{2n}^2 - 4 = 4F_{2n+1}F_{2n-1} + (F_{2n+1} - F_{2n-1})^2 = (F_{2n+1} + F_{2n-1})^2.
\]

Hence, the remaining two roots of \( P(Q(K_{p_1,p_2,p_3}), x) \) are integers equal to

\[
x_1 = \frac{3F_{2n}^2 - 2 - F_{2n}(F_{2n+1} + F_{2n-1})}{2} = \frac{3F_{2n}^2 - 2 - F_{2n}(2F_{2n} + F_{2n-1})}{2} = F_{2n}(F_{2n} - F_{2n-1}) - 1 = F_{2n}F_{2n-2} - 1,
\]

and

\[
x_2 = \frac{3F_{2n}^2 - 2 + F_{2n}(F_{2n+1} + F_{2n-1})}{2} = \frac{3F_{2n}^2 - 2 + F_{2n}(2F_{2n+1} - F_{2n})}{2} = F_{2n}(F_{2n} + F_{2n+1}) - 1 = F_{2n}F_{2n+2} - 1.
\]

\( \square \)
4. \( Q \)-integral complete multipartite graphs with \( s \geq 4 \)

We performed a computer search for \( Q \)-integral graphs \( K_{p_1,p_2,p_3,p_4} \), with \( p_1 < p_2 < p_3 < p_4, p_1 \leq 100 \) and \( p_4 \leq 1000, \) and found 46 such graphs. In Table 3 we list the parameters and \( Q \)-spectrum of those graphs, whose parameters are not the multiples of parameters of smaller such \( Q \)-integral graphs.

Table 3: The parameters and \( Q \)-spectrum of some \( Q \)-integral graphs \( K_{p_1,p_2,p_3,p_4} \). The exponents in parentheses denote eigenvalue multiplicities.

<table>
<thead>
<tr>
<th>( p_1 )</th>
<th>( p_2 )</th>
<th>( p_3 )</th>
<th>( p_4 )</th>
<th>( Q )-spectrum of ( K_{p_1,p_2,p_3,p_4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>15</td>
<td>22</td>
<td>36</td>
<td>{19, 43^{36}, 57^{34}, 63, 64^{14}, 73^2, 112}</td>
</tr>
<tr>
<td>9</td>
<td>20</td>
<td>45</td>
<td>100</td>
<td>{24, 74^{19}, 114, 129^{24}, 150, 154^{15}, 165^8, 234}</td>
</tr>
<tr>
<td>15</td>
<td>25</td>
<td>36</td>
<td>60</td>
<td>{36, 76^{16}, 106^{11}, 111^{24}, 121^{14}, 196}</td>
</tr>
<tr>
<td>24</td>
<td>49</td>
<td>144</td>
<td>294</td>
<td>{63, 212^{96}, 343, 367^{143}, 448, 462^{45}, 487^{23}, 679}</td>
</tr>
<tr>
<td>30</td>
<td>81</td>
<td>100</td>
<td>270</td>
<td>{81, 211^{269}, 301, 381^{99}, 400^{84}, 451^{29}, 661}</td>
</tr>
<tr>
<td>45</td>
<td>253</td>
<td>276</td>
<td>483</td>
<td>{252, 529, 574^{482}, 781^{275}, 804^{-252}, 919, 1741, 1012^{44}}</td>
</tr>
<tr>
<td>48</td>
<td>88</td>
<td>198</td>
<td>363</td>
<td>{121, 343^{166}, 433, 499^{197}, 576, 609^{87}, 649^{47}, 961}</td>
</tr>
<tr>
<td>51</td>
<td>144</td>
<td>289</td>
<td>816</td>
<td>{144, 484^{1014}, 892, 1011^{288}, 1156^{144}, 1249^{50}, 1708}</td>
</tr>
<tr>
<td>55</td>
<td>90</td>
<td>132</td>
<td>330</td>
<td>{112, 277^{129}, 387, 475^{132}, 517^{89}, 552^{54}, 847}</td>
</tr>
<tr>
<td>80</td>
<td>98</td>
<td>200</td>
<td>245</td>
<td>{175, 343, 376^{244}, 423^{199}, 448, 525^{97}, 543^{79}, 903}</td>
</tr>
<tr>
<td>98</td>
<td>175</td>
<td>252</td>
<td>450</td>
<td>{240, 525^{449}, 555, 723^{251}, 735, 800^{174}, 877^{97}, 1395}</td>
</tr>
</tbody>
</table>

A careful observation of these parameter sets shows that \( \{15, 25, 36, 60\}, \{30, 81, 100, 270\} \) and \( \{51, 144, 289, 816\} \) are all instances of \( 3a, a^2, 9b^2, 3ab^2 \) for particular values of \( a, b \in \mathbb{Z} \).

Let \( n = 3a + a^2 + 9b^2 + 3ab^2 \). The \( Q \)-characteristic polynomial of \( K_{3a,a^2,9b^2,3ab^2} \), by (1), has the form

\[
P(Q(K_{3a,a^2,9b^2,3ab^2}), x) = (x-n+3a)^{3a-1}(x-n+a^2)^{a^2-1}(x-n+9b^2)^{9b^2-1}(x-n+3ab^2)^{3ab^2-1}B(x),
\]

where

\[
B(x) = (x-n+6a)(x-n+2a^2)(x-n+18b^2)(x-n+6ab^2)
\]

\[
\left(1 - \frac{3a}{x-n+6a} - \frac{a^2}{x-n+2a^2} - \frac{9b^2}{x-n+18b^2} - \frac{3ab^2}{x-n+6ab^2}\right),
\]

which, after replacing \( n = 3a + a^2 + 9b^2 + 3ab^2 \), becomes

\[
B(x) = (x - (9b^2 + 3a(b - 1)^2 + a^2))(x - (9b^2 + 3a(b + 1)^2 + a^2))
\]

\[
(\text{a}^2 - (a+3)(a+3b^2)x + 36a^2b^2).
\]

Hence, \( K_{3a,a^2,9b^2,3ab^2} \) is \( Q \)-integral if and only if the discriminant of the quadratic factor above

\[
(a+3)^2(a+3b^2)^2 - 144a^2b^2 = [(a+3)(a+3b^2) - 12ab] [(a+3)(a+3b^2) + 12ab]
\]

\[
= [(2a - 3b^2 - 3a(a - (b^2 + 1))] [(2a + 3b^2) - 3a(a - (b^2 + 1))]
\]

is a perfect square.
The simplest way to achieve this is to set \( a = b^2 + 1 \)—this holds for the parameter sets \( \{15, 25, 36, 60\}, \{30, 81, 100, 270\} \) and \( \{51, 144, 289, 816\} \) as well. For \( a = b^2 + 1 \), \( B(x) \) becomes

\[
B(x) = (x - 9b^2) \left( x - 4 \left( b^2 + 1 \right)^2 \right) \\
\quad \cdot (x - (4b^4 - 6b^3 + 17b^2 - 6b + 4)) \left( x - (4b^4 + 6b^3 + 17b^2 + 6b + 4) \right).
\]

Hence, we proved

**Theorem 8.** The complete 4-partite graph \( K_{3(b^2 + 1), (b^2 + 1)^2, 9b^2, 3b^2(b^2 + 1)} \) is \( Q \)-integral for any \( b \in \mathbb{Z} \).

Further computer searches unearthed several more \( Q \)-integral complete multipartite graphs with \( s \geq 4 \). First, in Table 4 we list the parameters of seven \( Q \)-integral graphs \( K_{a_1, p_1, a_2, p_2, a_3, p_3, a_4, p_4} \), where not all \( a_i \)’s are ones and which have at most 120 vertices. Table 4 also contains nontrivial roots of the \( Q \)-characteristic polynomial (1), i.e., the roots of the factor

\[
\prod_{i=1}^{r} (x - n + 2p_i) \left( 1 - \sum_{i=1}^{r} \frac{p_i}{x - n + 2pi} \right)
\]

Second, in Table 5 we list the parameters and nontrivial part of the \( Q \)-spectrum for seven \( Q \)-integral graphs \( K_{a_1, p_1, a_2, p_2, a_3, p_3, a_4, p_4, a_5, p_5} \) on at most 1000 vertices.

**Table 4:** The parameters of \( Q \)-integral graphs \( K_{a_1, p_1, a_2, p_2, a_3, p_3, a_4, p_4} \).

<table>
<thead>
<tr>
<th>( a_1 )</th>
<th>( p_1 )</th>
<th>( a_2 )</th>
<th>( p_2 )</th>
<th>( a_3 )</th>
<th>( p_3 )</th>
<th>( a_4 )</th>
<th>( p_4 )</th>
<th>Nontrivial part of the ( Q )-spectrum</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>1</td>
<td>6</td>
<td>1</td>
<td>9</td>
<td>1</td>
<td>24</td>
<td>[71, 39, 32, 11]</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>2</td>
<td>9</td>
<td>1</td>
<td>15</td>
<td>1</td>
<td>20</td>
<td>[117, 59, 44, 33]</td>
</tr>
<tr>
<td>17</td>
<td>2</td>
<td>2</td>
<td>7</td>
<td>2</td>
<td>9</td>
<td>1</td>
<td>11</td>
<td>[143, 67, 61, 56]</td>
</tr>
<tr>
<td>20</td>
<td>1</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>7</td>
<td>2</td>
<td>10</td>
<td>[151, 76, 69, 63]</td>
</tr>
<tr>
<td>18</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>7</td>
<td>2</td>
<td>22</td>
<td>[161, 87, 80, 61]</td>
</tr>
<tr>
<td>22</td>
<td>1</td>
<td>4</td>
<td>10</td>
<td>1</td>
<td>14</td>
<td>1</td>
<td>19</td>
<td>[171, 87, 69, 60]</td>
</tr>
<tr>
<td>17</td>
<td>3</td>
<td>2</td>
<td>7</td>
<td>1</td>
<td>12</td>
<td>2</td>
<td>18</td>
<td>[209, 101, 92, 83]</td>
</tr>
</tbody>
</table>

**Table 5:** The parameters of \( Q \)-integral graphs \( K_{a_1, p_1, a_2, p_2, a_3, p_3, a_4, p_4, a_5, p_5} \).

<table>
<thead>
<tr>
<th>( a_1 )</th>
<th>( p_1 )</th>
<th>( a_2 )</th>
<th>( p_2 )</th>
<th>( a_3 )</th>
<th>( p_3 )</th>
<th>( a_4 )</th>
<th>( p_4 )</th>
<th>( a_5 )</th>
<th>( p_5 )</th>
<th>Nontrivial part of the ( Q )-spectrum</th>
</tr>
</thead>
<tbody>
<tr>
<td>39</td>
<td>3</td>
<td>3</td>
<td>12</td>
<td>2</td>
<td>17</td>
<td>1</td>
<td>27</td>
<td>3</td>
<td>45</td>
<td>[655, 331, 319, 301, 280]</td>
</tr>
<tr>
<td>46</td>
<td>2</td>
<td>9</td>
<td>10</td>
<td>7</td>
<td>15</td>
<td>6</td>
<td>21</td>
<td>4</td>
<td>27</td>
<td>[1011, 512, 497, 485, 471]</td>
</tr>
<tr>
<td>15</td>
<td>4</td>
<td>18</td>
<td>10</td>
<td>4</td>
<td>13</td>
<td>5</td>
<td>15</td>
<td>9</td>
<td>20</td>
<td>[1067, 537, 523, 519, 512]</td>
</tr>
<tr>
<td>18</td>
<td>11</td>
<td>2</td>
<td>18</td>
<td>6</td>
<td>20</td>
<td>3</td>
<td>26</td>
<td>5</td>
<td>37</td>
<td>[1189, 587, 580, 569, 553]</td>
</tr>
<tr>
<td>36</td>
<td>5</td>
<td>13</td>
<td>15</td>
<td>6</td>
<td>25</td>
<td>3</td>
<td>39</td>
<td>1</td>
<td>49</td>
<td>[1341, 673, 649, 621, 596]</td>
</tr>
<tr>
<td>9</td>
<td>8</td>
<td>4</td>
<td>14</td>
<td>5</td>
<td>19</td>
<td>14</td>
<td>23</td>
<td>3</td>
<td>50</td>
<td>[1339, 676, 665, 655, 607]</td>
</tr>
<tr>
<td>20</td>
<td>2</td>
<td>17</td>
<td>9</td>
<td>2</td>
<td>16</td>
<td>1</td>
<td>29</td>
<td>12</td>
<td>42</td>
<td>[1454, 752, 730, 718, 698]</td>
</tr>
</tbody>
</table>
Third, we found an example of \(Q\)-integral complete multipartite graph with \(s = 6\) as well:

\[
K_{\overline{4}, 4, 6}, \overline{10, 7}, (10, 13), 50, 19, 25, 24, 53, 33
\]

which has \([9847, 4932, 4921, 4915, 4901, 4889]\) as the nontrivial part of its \(Q\)-spectrum.

5. Conclusions

We can now see that both questions of Zhao et al. [13] have affirmative answers: Question 4.1 is answered by Theorem 8 and examples from Section 4, while Question 4.2 is answered by Theorems 7 and 8 and examples from Tables 1 and 3. All these results suggest that \(Q\)-integral complete multipartite graphs are likely to exist for an arbitrarily large \(s\), regardless of the condition that all \(a_i\)’s be equal to one.

At the end, we have to note that the same authors have published another paper [14] in which they studied the Seidel spectrum of the complete multipartite graphs, where the Seidel matrix \(S(G)\) of a graph \(G\) is defined as \(S(G) = J - I - 2A(G)\), with \(J\) and \(I\) being the all-one and the unit matrix, respectively. In particular, they obtained that the Seidel characteristic polynomial of \(K_{a_1, \ldots, a_s}^{p_1, \ldots, p_r}\) is

\[
P(S(K_{p_1, \ldots, p_r}), x) = (x + 1)^{n-r} \prod_{i=1}^{r} (x - 2p_i + 1) \left( 1 + \sum_{i=1}^{r} \frac{p_i}{x - 2p_i + 1} \right).
\]

After deriving results about Seidel integral complete multipartite graphs, analogous to those in [13], the authors finished the manuscript with analogous and equally numbered Questions 4.1 and 4.2 in [14]:

**Question 4.1** [14] Are there any Seidel integral complete multipartite graphs \(K_{a_1, p'_1, \ldots, a_s, p'_s}\) for arbitrarily large \(s\)?

**Question 4.2** [14] Are there any Seidel integral complete multipartite graphs \(K_{a_1, p'_1, \ldots, a_s, p'_s}\) with \(a_1 = \cdots = a_s = 1\) when \(s \geq 3\)?

It is straightforward to see that for the nontrivial factors of the \(Q\)- and Seidel characteristic polynomials

\[
Q^*(x) = \frac{P(Q(K_{p_1, \ldots, p_r}), x)}{\prod_{i=1}^{r} (x - n + p_i + 1)} = \prod_{i=1}^{r} (x - n + 2p_i) \left( 1 + \sum_{i=1}^{r} \frac{p_i}{x - n + 2p_i} \right)
\]

and

\[
S^*(x) = \frac{P(S(K_{p_1, \ldots, p_r}), x)}{(x + 1)^{n-r}} = \prod_{i=1}^{r} (x - 2p_i + 1) \left( 1 + \sum_{i=1}^{r} \frac{p_i}{x - 2p_i + 1} \right)
\]

holds that

\[
S^*(x) = (-1)^r Q^*(n-x-1).
\]
Hence, $S^*(x)$ has integer roots if and only if $Q^*(x)$ has integer roots or, in other words, a complete multipartite graph is Seidel integral if and only if it is $Q$-integral. Therefore, our results from previous sections give a partially affirmative answer to Question 4.1 and an affirmative answer to Question 4.2 from [14] as well.

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References


