Laplacian–Energy–Like Invariant: 
Laplacian Coefficients, Extremal Graphs and Bounds

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1 Introduction

Let \( G = (V, E) \) be a simple graph with \( n = |V| \) vertices and \( m = |E| \) edges. Let \( A \) be the adjacency matrix of \( G \), and let the eigenvalues of \( A \) be let

\[ \lambda_1 \geq \cdots \geq \lambda_n. \]

The energy of a graph \( G \) was introduced by Gutman [18] in 1978, as an approximation of the total \( \pi \)-electron energy within the Hückel molecular orbital model, and defined as

\[ E(G) = \sum_{i=1}^{n} |\lambda_i|. \]
It was a considerably studied descriptor in chemistry, but was mostly being overlooked by mathematicians until 2000s. Early mathematical results on graph energy were reviewed by Gutman in [19], and the joint appearance of this survey and the first important mathematical result (together with a conjecture) on the maximum graph energy by Koolen and Moulton [37] may have been the reason for an ever increasing interest of mathematicians in graph energy since 2001.

Further impetus to this interest were independent observations by several researchers that the energy may be defined not only for adjacency matrices of graphs, but for arbitrary matrices in general. One approach to this generalization was used by Nikiforov in his proof of the Koolen–Moulton conjecture [50], where he defined the energy of a matrix as a sum of its singular values. Another approach followed the definition of Laplacian energy by Gutman and Zhou [25]. Let

\[ \mu_1 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0 \]

be the eigenvalues of the Laplacian matrix \( L = D - A \) of \( G \), where \( D \) is the diagonal matrix of vertex degrees. The Laplacian energy of a graph \( G \) is then defined as

\[ LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|. \quad (2) \]

Noting that the average value of the eigenvalues of \( A \) is 0, while the average value of the eigenvalues of \( L \) is \( \frac{2m}{n} \), which explains the term \( \frac{2m}{n} \) in the definition of LE. It soon became clear [57] that the energy of an arbitrary diagonalizable matrix may be defined as the absolute deviation of its eigenvalues from their average value. A number of results on different graph energies (such as the signless Laplacian energy, distance energy, Harary energy, Randić energy, etc.) then poured in the last 5-10 years, to which other chapters of this book are devoted.

However, these energies do not always have the properties that the graph energy has. For example, unlike the graph energy, the Laplacian energy is not additive with respect to the connected components of the graph. In order to overcome this, Liu and Liu [40] introduced the Laplacian–energy–like invariant, shortly \( LEL \), defined as

\[ LEL(G) = \sum_{i=1}^{n} \sqrt{|\mu_i|}. \quad (3) \]

The \( LEL \) was proved to be a useful molecular descriptor already. In [63], the \( LEL \) and several other graph invariants/topological indices were compared with physical proper-
ties of octanes and a second data set of polycyclic aromatic hydrocarbons. The \( LEL \) performed very well and showed the highest correlation coefficient with the boiling point.

In addition, the \( LEL \) turns out to be a special case of Nikiforov’s generalization of graph energy. Let \( G \) be given an arbitrary orientation \( \sigma \) of its edges, and let \( I^\sigma \) denote the vertex-edge incidence matrix defined as

\[
I^\sigma_{v,e} = \begin{cases} 
1, & \text{if } v \text{ is the head of } e \\
-1, & \text{if } v \text{ is the tail of } e \\
0, & \text{otherwise.}
\end{cases}
\]

Then \( L = I^\sigma I^{\sigma T} \), so that the square roots of the Laplacian eigenvalues of \( G \) represent the singular values of \( I^\sigma \). Hence the \( LEL \) can be viewed as the incidence energy of a graph with an arbitrary orientation in Nikiforov’s sense.

If one takes an unoriented incidence matrix instead, one ends up with the incidence energy, which is defined as

\[
IE(G) = \sum_{i=1}^{n} \sqrt{\nu_i}
\]

where the \( \nu_i \) are the eigenvalues of the signless Laplacian \( Q = D + A \). Another chapter of this book is devoted to the incidence energy. In the case of a bipartite graph, however, the spectra of Laplacian and signless Laplacian coincide [47, 48], and one has \( LEL(G) = IE(G) \). This coincidence was first observed in [21] and [22].

The literature on properties of the invariant \( LEL \) is rapidly growing; Liu, Huang and You [39] provided a first survey. The main aim of this chapter is to give an updated survey including many newer results and to put them into perspective.

In Section 2 we start with the fundamental relation between the \( LEL \) and the coefficients of the Laplacian characteristic polynomial. In Section 3 we present some graph transformations that simultaneously increase or decrease these coefficients, and survey a number of results describing graphs extremal with respect to the \( LEL \) in various classes of graphs. In Section 4 we provide an overview of different bounds on the \( LEL \), starting first with bounds that depend on the numbers of vertices and edges only, and slowly increasing in the complexity of their expressions. Finally, in Section 5 we list various other results on the \( LEL \) that do not clearly fit in any of the previous sections.
2 Coefficients of the Laplacian Characteristic Polynomial

The Laplacian coefficients are the coefficients \( c_i(G) \) in the expansion of the characteristic polynomial of the Laplacian matrix \( L_G \) of \( G \):

\[
\det(xI - L_G) = \sum_{i=0}^{n} (-1)^i c_i(G) x^{n-i}.
\]

The signless Laplacian coefficients, on the other hand, are the coefficients \( \phi_i(G) \) in the expansion of the characteristic polynomial of the signless Laplacian \( Q_G \):

\[
\det(xI - Q_G) = \sum_{i=0}^{n} (-1)^i \phi_i(G) x^{n-i}.
\]

Let us recall here that the Laplacian of \( G \) is \( L_G = D_G - A_G \), while the signless Laplacian matrix of a graph \( G \) is defined as \( Q_G = D_G + A_G \) [11].

The first three coefficients of the Laplacian and the signless Laplacian characteristic polynomials coincide, and they are equal to

\[
1, \quad -2m, \quad 2m^2 - m - \frac{1}{2} \sum_{i=1}^{n} d_i^2
\]

respectively, where \( n \) is as usual the number of vertices, \( m \) the number of edges and \( d_1, d_2, \ldots, d_n \) the vertex degrees. A relatively complicated expression for the fourth Laplacian coefficient was found by Oliveira et al. [52], and for the fourth signless Laplacian coefficient by Wang et al. [71].

Akbari et al. [1] have shown that \( \phi_i(G) \geq c_i(G) \) for each \( i = 0, \ldots, n \), and consequently that

\[
IE(G) \geq LEL(G). \quad (4)
\]

This can also be seen as a consequence of Theorem 1 below. As mentioned earlier, equality holds if (and in fact only if) \( G \) is bipartite.

Generally, the Laplacian coefficient \( c_i(G) \) is equal to the number of rooted spanning forests of \( G \) with \( i \) components (each of the components is equipped with its own root; the number of ways to assign roots to a specific spanning forest with \( i \) components is therefore the product of the component sizes), see for instance [4]. This also shows immediately that the coefficients are monotone under addition/deletion of edges.

Many results on the \( LEL \) are based on its connection to the Laplacian coefficients. The following key result was first stated in [59] and a corrected proof was given in [33].
Theorem 1. [33,59] The LEL is an increasing function in all coefficients of the Laplacian characteristic polynomial; in particular, if the Laplacian characteristic polynomials of two graphs $G$ and $G'$ are $\sum_{j=0}^{n}(-1)^{j}c_{j}(G)x^{n-j}$ and $\sum_{j=0}^{n}(-1)^{j}c_{j}(G')x^{n-j}$ respectively, and the coefficients satisfy $c_{j}(G) \geq c_{j}(G')$ for all $j$, then $LEL(G) \geq LEL(G')$. The inequality is strict if at least one of the inequalities $c_{j}(G) \geq c_{j}(G')$ holds with strict inequality.

Since this theorem is central for the theory of the $LEL$ (almost all extremal results in the following section are derived from it), we sketch its proof. By Vieta’s formulas, the coefficients $c_{j}$ are the elementary symmetric polynomials in the roots $\mu_{1}, \mu_{2}, \ldots, \mu_{n}$:

$$c_{j} = \sum_{J \subseteq \{1,2,\ldots,n\} \atop |J|=j} \prod_{i \in J} \mu_{i}.$$  

Now we consider the function $F$ that maps the $n$-tuple $(\mu_{1}, \mu_{2}, \ldots, \mu_{n})$ to the $n$-tuple $(c_{1}, c_{2}, \ldots, c_{n})$ as a function from (some subset of) $\mathbb{R}^{n}$ to $\mathbb{R}^{n}$. In other words, the $j$-th coordinate of $F(x_{1}, x_{2}, \ldots, x_{n})$ is given by

$$F_{j}(x_{1}, x_{2}, \ldots, x_{n}) = \sum_{J \subseteq \{1,2,\ldots,n\} \atop |J|=j} \prod_{i \in J} x_{i}.$$  

In order to find the partial derivatives $\partial LEL/\partial c_{j}$, the Jacobian of the inverse $F^{-1}$ is required. Ilić, Krtinić and Ilić proved in [33] that this Jacobian matrix is given by

$$J_{F^{-1}} = \left[ (-1)^{j-1} \frac{x_{i}^{n-j}}{\omega'(x_{i})} \right]_{i,j=1,2,\ldots,n} = \begin{bmatrix} x_{1}^{n-2} & -x_{1}^{n-2} & \cdots & (-1)^{n-1} & -1 \\ \omega'(x_{1}) & \omega'(x_{1}) & \cdots & (-1)^{n-1} & -1 \\ x_{2}^{n-2} & -x_{2}^{n-2} & \cdots & (-1)^{n-1} & -1 \\ \omega'(x_{2}) & \omega'(x_{2}) & \cdots & (-1)^{n-1} & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n}^{n-2} & -x_{n}^{n-2} & \cdots & (-1)^{n-1} & -1 \\ \omega'(x_{n}) & \omega'(x_{n}) & \cdots & (-1)^{n-1} & -1 \end{bmatrix}$$

where $\omega(x) = \prod_{i=1}^{n}(x-x_{i})$. This is done by first determining the Jacobian $J_{F}$ and proving that the product of the two matrices is indeed the identity matrix. The formula for the Jacobian implies that

$$\frac{\partial LEL}{\partial c_{j}} = \sum_{i=1}^{n} \frac{\partial LEL}{\partial \mu_{i}} \cdot \frac{\partial \mu_{i}}{\partial c_{j}} = \sum_{i=1}^{n} \frac{1}{2\sqrt{\mu_{i}}} \cdot \frac{(-1)^{j-1} \mu_{i}^{n-j}}{\omega'(\mu_{i})} = \frac{(-1)^{j-1}}{2} \sum_{i=1}^{n} \frac{\mu_{i}^{n-j-1/2}}{\omega'(\mu_{i})}$$

where $\omega(x)$ is now defined as $\omega(x) = \prod_{i=1}^{n}(x-x_{i})$. We set $f(x) = x^{n-j-1/2}$ and notice that

$$P(x) = \sum_{i=1}^{n} f(\mu_{i}) \cdot \frac{\prod_{k \neq i}(x-\mu_{k})}{\omega'(\mu_{i})}.$$
is the Lagrange interpolation polynomial of the function $f(x)$ with respect to the points $\mu_1, \mu_2, \ldots, \mu_n$. The sum

$$S = \sum_{i=1}^{n} \frac{\mu_i^{n-j-1/2}}{\omega'(\mu_i)}$$

is exactly the leading coefficient (coefficient of $x^{n-1}$) of this polynomial, which is also equal to $(n-1)!P^{(n-1)}(x)$ for any $x$. Since $P(x) - f(x)$ has $n$ real zeros, it follows by iterated application of Rolle’s theorem that $P^{(n-1)}(x) - f^{n-1}(x)$ has a real root as well, which must be contained in the interval between the smallest and the largest of the $\mu_i$.

So we have, for some $\xi > 0$,

$$S = (n-1)!P^{(n-1)}(\xi) = (n-1)!f^{(n-1)}(\xi).$$

Now since

$$f^{(n-1)}(\xi) = \prod_{r=1}^{n-1} (n-j+1/2-r) \cdot \xi^{1/2-j}$$

has exactly $j-1$ negative factors (corresponding to $r = n-j+1, \ldots, n-1$), the sign of $S$ is $(-1)^{j-1}$, which means that the derivative

$$\frac{\partial LEL}{\partial c_j} = \frac{(-1)^{j-1}}{2} \cdot S$$

is indeed positive for all $j$. Thus $LEL(G)$ is an increasing function of each of the coefficients $c_j$, which is what we wanted to prove. Some additional care is needed to deal with the situation that some of the $\mu_i$ are equal, which is done by continuity.

The statement of Theorem 1 can be expressed in terms of a partial order of graphs by their Laplacian coefficients. We write $G \succeq G'$ if $c_j(G) \geq c_j(G')$ for all $j$. Then Theorem 1 simply becomes

$$G \succeq G' \iff LEL(G) \geq LEL(G')$$

and this can be exploited to obtain a number of extremal results, which will be presented in the following section.

### 3 Extremal Values

One of the most natural questions in connection with any graph invariant is to ask for its extremal values in different classes of graphs and the associated extremal graphs. The $LEL$ is not an exception here, and a large number of papers deals with questions of
this type. An important tool in this regard is the connection to the coefficients of the Laplacian characteristic polynomials, specifically Theorem 1, which is the key to almost all extremal results for the $LEL$.

The approach to determine the graph in a specified class that maximizes or minimizes the $LEL$ is simply to find a graph that simultaneously maximizes or minimizes all Laplacian coefficients. Of course, such a graph need not necessarily exist, but in a great number of instances it does, as will become apparent from the following list of extremal results. Once such a graph has been found, it must be extremal in view of Theorem 1.

The main tool to prove that a graph has the property of maximizing or minimizing all Laplacian coefficients is to apply transformations that are guaranteed to increase or decrease all coefficients simultaneously (some of them may remain the same).

### 3.1 Some useful transformations

The archetype of graph transformations that simultaneously increase or decrease all Laplacian coefficients are the $\pi$- and $\sigma$-transformations due to Mohar [49]. We will focus on these two transformations and their generalizations. They were originally designed to be used in the context of trees, but they extend to other graphs as well.

**The $\pi$-transformation** A pendent path $v_0v_1\ldots v_k$ in a tree is a path with the property that $v_k$ is a leaf and $v_1, v_2, \ldots, v_{k-1}$ all have degree 2. The $\pi$-transformation takes two pendent paths $v_0v_1\ldots v_k$ and $u_0u_1\ldots u_\ell$ with a common end $v_0 = u_0$ whose degree is at least 3 (otherwise, the tree would be a path) and replaces them by a new path $v_0v_1\ldots v_{k+\ell}$, see Figure 1. If a tree $T$ is transformed in this way to yield a tree $T'$, then $T' \succeq T$, and Mohar [49] even provided explicit lower bounds for the differences $c_j(T') - c_j(T)$.

Ilić [29] considered the generalized situation that the two paths are replaced by paths of lengths $k'$ and $\ell'$ (such that the total number of vertices is preserved, i.e. $k' + \ell' = k + \ell$) and showed that the Laplacian coefficients are nondecreasing as the length of the longer path increases. This was further generalized [66, 67] to the case that the two paths are attached to distinct vertices that are at a fixed distance from each other.
The $\sigma$-transformation Suppose that a vertex $v$ in a tree $T$ has only one neighbor $u$ that is not a leaf, while all other neighbors $u_1, u_2, \ldots, u_k$ are leaves. The $\sigma$-transform removes the edges $vu_1, vu_2, \ldots, vu_k$ and replaces them by edges $uu_1, uu_2, \ldots, uu_k$ (Figure 2). If a tree $T'$ is obtained by a $\sigma$-transform from a tree $T$, then $T' \preceq T$, and again explicit lower bounds for the difference between Laplacian coefficients are provided in [49].

The $\sigma$-transformation was also generalized in many different ways: Ilić [29] extended it to the situation that the pendent edges $vu_1, \ldots, vu_k$ are pendent paths (see also [30,32,66]) and called it the $\delta$-transformation in this case. Both are in fact a special case of what is called $\alpha$-transformation in [27,28]: here, $u$ and $v$ can be neighbors in an arbitrary graph, provided that the edge $uv$ is not part of a triangle. The transform consists of merging $u$ and $v$ and attaching a new pendent edge (and vertex) to the merged vertex. This in turn is a special case of the $\chi_s$-transform as defined in [67], which takes any path $uw_1w_2\ldots w_{s-1}v$, where all internal vertices have degree 2 and the path (of length $s$) is not part of a cycle of length $s$ or $s + 1$, then deletes the edge $w_{s-1}v$, merges the vertices $u$ and $v$ and adds an additional vertex $w_s$ and edge $w_{s-1}w_s$ at the end of the path. For trees, Csikvári [10] built a poset structure on the set of all trees of given order around this transformation (which he called a generalized tree shift). See Figure 3 for an illustration.
There are several further transformations that appear in the literature (see [27,28,53,62,64,65]), but most of them are more specialized to deal with specific types of graphs, such as unicyclic graphs, so we refrain from a comprehensive discussion.

3.2 Trees

The first extremal result derived from Theorem 1 is concerned with trees:

**Theorem 2.** [59] *For a tree $T$ of order $n$, the inequalities

$$LEL(S_n) \leq LEL(T) \leq LEL(P_n)$$

hold, with equality if and only if $T$ is the star $S_n$ or the path $P_n$ respectively.*

The proof is based on a result of Zhou and Gutman [79] that the star simultaneously minimizes all Laplacian coefficients while the path simultaneously maximizes them among trees of given order (generalizing earlier results [24] that only covered a few coefficients). To show that they are indeed unique as extremal graphs, the aforementioned transformations due to Mohar can be used to show that strict inequality holds for some of the coefficients.

Trees with second- and third-largest as well as second- and third-smallest $LEL$ were determined as well [68] (these results could partly also be inferred from those in [76], where only the coefficients of the Laplacian characteristic polynomial are considered). Naturally, their structures are very close to paths and stars.

In the aforementioned paper [59], a result on trees with given maximum degree was proven as well, albeit only for the maximum:
Theorem 3. [59] Among trees of order \( n \) and maximum degree \( \Delta \), the unique tree that maximizes the LEL is obtained by attaching a path of \( n - \Delta \) vertices to the center of a star of order \( \Delta \).

Several other different classes of trees were investigated as well. Ilić and Ilić [32] provided the following results on trees with a given number of leaves or vertices of degree 2:

Theorem 4. [32] Among trees of order \( n \) with exactly \( k \) leaves, the unique tree that minimizes the LEL is the balanced starlike tree obtained by attaching \( k \) paths, all of length \( \lfloor (n - 1)/k \rfloor \) or \( \lceil (n - 1)/k \rceil \), to a common center. This balanced starlike tree is also the unique tree with \( n \) vertices and \( n - k - 1 \) vertices of degree 2 that minimizes the LEL.

It is known that among trees of order \( n \) with \( k \) leaves, the Wiener index (which coincides with the coefficient of \( x^2 \) in the Laplacian characteristic polynomial) is maximized by a dumbbell (a path with \( \lfloor k/2 \rfloor \) and \( \lceil k/2 \rceil \) vertices attached to the two ends); however, this tree does not maximize all other coefficients, so an analogous result for the LEL has not been proven yet and might in fact be quite difficult to obtain.

Trees and graphs with given diameter or radius were investigated in [31,76], with the following results:

Theorem 5. [31,76] The unique tree of order \( n \) with diameter \( d \) that minimizes the LEL is obtained by attaching \( n - d - 1 \) leaves to the central vertex (one of the two central vertices if \( d \) is odd) of a path. The unique tree (and in fact the unique connected graph) of order \( n \) with radius \( r \) that minimizes the LEL is obtained by attaching \( n - 2r - 1 \) leaves to the central vertex of a path of length \( 2r \).

Again, the analogous problem for the maximum seems to be more intricate. Partial results on the ordering of trees with diameter 3 or 4 with respect to their Laplacian coefficients can be found in [76]. Specifically, the tree with diameter 3 and maximum LEL is the balanced double-star (for which the degrees of the two non-leaves differ by at most 1).

Several articles deal with trees whose matching number is fixed as well as the special case of trees with a perfect matching. The following theorem was first proved by Ilić in [30], and an alternative proof was given by He and Li [28].
Theorem 6. [28,30] Among trees with \( n \) vertices and matching number \( k \), the minimum of the LEL is attained by a spur, i.e., the tree obtained by attaching an additional leaf to \( k - 1 \) leaves of a star of order \( n - k + 1 \), and the spur is unique with this property.

The special case of trees with a perfect matching was also considered earlier in [29], as well as in [28] and [66]. The tree with maximum LEL is obviously the path in this case. The trees with second-and third-largest and second- and third-smallest LEL were also characterized in [28] and [66].

Finally, we mention a result of Lin and Yan on trees with a given bipartition, i.e., given sizes of the two sets in the unique bipartition of the vertices of a tree.

Theorem 7. [38] Among all trees with bipartition \((p,q)\), the unique tree that minimizes the LEL is the double star obtained by joining the centers of two stars of order \( p \) and \( q \) respectively by an edge.

The tree with second-smallest LEL among trees with bipartition \((p,q)\) was also characterized in [38]. The analogous problem for the maximum seems to be more difficult, as it appears to be the case for most other classes of graphs as well.

3.3 Unicyclic, bicyclic and tricyclic graphs

Once the extremal trees have been determined, graphs with small cyclomatic number are usually the next step. For unicyclic graphs, we have the following result:

Theorem 8. [62] Among unicyclic graphs of given order, the minimum of the LEL is attained by a star with an additional edge; the maximum of the LEL is attained by the cycle. Both graphs are unique with this property.

The minimum of the LEL was further investigated for bicyclic and tricyclic graphs. As one might expect, the extremal graph is obtained by adding further edges to the extremal unicyclic graph:

Theorem 9. [27,54] The minimum of the LEL among bicyclic graphs of given order \( n \) is uniquely attained by a graph constructed from a star by adding two additional edges with a common vertex (equivalently, constructed from a complete graph \( K_4 \) by removing an edge and attaching \( n - 4 \) pendent vertices to a vertex of degree 3). The minimum of the LEL among tricyclic graphs of given order \( n \) is uniquely attained by the graph constructed from a \( K_4 \) by attaching \( n - 4 \) pendent vertices to one of its vertices.
Further restrictions were considered as well, as in the case of trees. We mention in particular unicyclic and bicyclic graphs with given matching number [64, 65], unicyclic graphs with given number of pendent vertices [53] and bipartite bicyclic graphs [9].

### 3.4 General graphs

Since the coefficients of the Laplacian and therefore also the LEL are monotone with respect to addition of edges, the minimum and maximum of the LEL among arbitrary graphs of order $n$ are attained by the empty graph $E_n$ (for which $LEL(E_n) = 0$) and the complete graph $K_n$ (for which $LEL(K_n) = (n - 1)\sqrt{n}$) respectively. Liu, Liu and Tan [45] went further and characterized the graphs with second-largest, third-largest, ..., ninth-largest LEL as well. Each of them is obtained by removing at most three edges from a complete graph.

If only connected graphs are considered, then the minimum is obviously attained by a tree, and we already know that the minimum in this case is $LEL(S_n) = (n - 2) + \sqrt{n}$ (cf. Theorem 13 in the following section).

Graphs with given matching number were considered in [74] and [75]. The minimum is again attained by a tree, so Theorem 6 applies here as well. For the maximum, on the other hand, there are different cases that need to be distinguished: the following theorem was proved by Xu and Das [74]; a partial version can also be found in [75].

**Theorem 10.** [74] Let $b$ be the largest root of the cubic equation

$$(2\sqrt{2} + 1)y^3 - (2 + \sqrt{n})y^2 - (n + 2\sqrt{2} - 1)y + n + \sqrt{n} = 0.$$ 

Let $G$ be a graph of order $n \geq 5$ with matching number $k > 2$.

1. If $k = [n/2]$, then $LEL(G) \leq (n - 1)\sqrt{n}$ with equality if and only if $G$ is a complete graph,

2. if $b^2 < k < [n/2]$, then $LEL(G) \leq \sqrt{n} + 2(k - 1)\sqrt{2} + n - 2k - k\sqrt{n}$ with equality if and only if $G \cong K_1 \cup (\overline{K_{n-2k}} \cup K_{2k-1})$ (here and in the following, $\cup$ denotes the join operation, which takes the disjoint union of two graphs and adds all possible edges between the two),

3. if $k = b^2$, then $LEL(G) \leq k\sqrt{n} + (n - k - 1)\sqrt{k}$ with equality if and only if $G \cong K_1 \cup (\overline{K_{n-2k}} \cup K_{2k-1})$ or $G \cong K_k \cup \overline{K_{n-k}}$. 
4. if $2 < k < b^2$, then $\text{LEL}(G) \leq k\sqrt{n} + (n - k - 1)\sqrt{k}$ with equality if and only if $G \cong K_k \bigvee K_{n-k}$.

Similar results were obtained for graphs with prescribed connectivity or chromatic number in [56] (although the results there are stated for the coefficients of the Laplacian characteristic polynomial only):

**Theorem 11.** [56]

1. The maximum of the LEL among graphs of order $n$ with chromatic number $\chi$ is attained by the complete multipartite graph with $\chi$ partite sets of size $\lfloor n/\chi \rfloor$ or $\lceil n/\chi \rceil$.

2. The maximum of the LEL among graphs of order $n$ whose connectivity (or edge connectivity) is $k$ is attained by the graph $(K_1 \cup K_{n-k-1}) \bigvee K_k$.

4 Bounds for the LEL

Since calculating spectral graph invariants requires the use of numerical algorithms, the goal of many researchers is to obtain bounds for them in terms of simpler invariants, most usually the numbers of vertices and edges. Many such bounds were found for the energy $E(G)$ earlier. Gutman et al. [26] observed that certain bounds for $\text{LEL}(G)$ follow from the corresponding bounds for $E(G)$, by noting that both $E(G)$ and $\text{LEL}(G)$ may be put in a similar framework. Suppose that for some numbers $q_1, \ldots, q_N$, which can be computed from a graph $G$ by some fixed procedure, we have

$$q_i \geq 0$$  \hspace{1cm} (6)

$$\sum_{i=1}^{n} q_i^2 = 2M$$  \hspace{1cm} (7)

$$\prod_{i=1}^{n} q_i = P.$$  \hspace{1cm} (8)

These three conditions then enable one to obtain bounds for the quantity $Q = \sum_{i=1}^{N} q_i$ in terms of $N, M,$ and $P$. If one sets $N = n$, $M = m$, and $q_i = |\lambda_i|$, then $P$ is equal to $|\det(A)|$ and the quantity $Q$ represents the energy $E(G)$, while if one sets $N = n-1$, $M = m$, and $q_i = \sqrt{\mu_i}$, then $P$ is equal to $\sqrt{n\mathcal{T}_G}$, where $t_G$ represents the number of spanning trees of $G$, and $Q$ represents the Laplacian-energy-like invariant $\text{LEL}(G)$. 


4.1 Bounds involving the numbers of vertices and edges only

The simplest bounds for $LEL(G)$ are expressed in terms of the numbers of vertices $n$ and edges $m$.

**Theorem 12.** [40] If $G$ is a connected graph, then

$$LEL(G) \leq \sqrt{2m(n-1)}$$

with equality if and only if $G \cong K_n$.

We provide a proof of this bound as a simple representative example. Since $\mu_n = 0$ and $\sum_{i=1}^{n-1} \mu_i = \sum_{i=1}^{n} \mu_i = 2m$, the arithmetic-quadratic mean inequality yields

$$\frac{LEL(G)}{n-1} = \frac{\sum_{i=1}^{n} \sqrt{\mu_i}}{n-1} = \frac{\sum_{i=1}^{n-1} \sqrt{\mu_i}}{n-1} \leq \sqrt{\frac{\sum_{i=1}^{n-1} \mu_i}{n-1}} = \sqrt{\frac{2m}{n-1}}$$

from which the desired bound follows.

Theorem 12 is the $LEL$-counterpart of the famous McClelland’s bound [46]

$$E(G) \leq \sqrt{2mn}.$$  

It is also a special case of a more general result of Liu and Liu [40]

$$LEL(G) \leq \sqrt{2m(n-p)}$$

where $p$ is the number of connected components of $G$, with equality if and only if each component of $G$ is either a complete graph of a fixed size or an isolated vertex. The proof is still the same, now exploiting the fact that the multiplicity of 0 as an eigenvalue of the Laplacian is exactly the number of components.

We have already seen that the complete graph $K_n$ has the greatest $LEL$ among graphs with $n$ vertices. For the minimal value of the $LEL$, we have the following:

**Theorem 13.** [40] For a connected graph $G$ with $n$ vertices,

$$(n-2) + \sqrt{n} \leq LEL(G)$$

with equality if and only if $G \cong S_n$.

This follows from the observation that the $LEL$ decreases when edges are removed together with the fact that the tree with minimal $LEL$ is a star.

The next bound depends only on the number of edges $m$.  

Theorem 14. [40] For a graph $G$ with $n$ vertices and $m$ edges,

$$\sqrt{2m} \leq LEL(G)$$

(11)

with equality if and only if $G \cong nK_1$ or $G \cong K_2 \cup (n-2)K_1$.

The following bound presents an improvement over (11) when a graph has at least $n/2$ edges.

Theorem 15. [22,39] For a graph $G$ with $n$ vertices and $m$ edges,

$$\frac{2m}{\sqrt{n}} \leq LEL(G)$$

(12)

with equality if and only if $G \cong K_n$ or $G \cong \overline{K_n}$.

The previous bound holds for the incidence energy as well (which implies the $LEL$ bound in view of (5)). Another incidence energy related bound improves upon (12) when the graph is bipartite.

Theorem 16. [22,77] If $G$ is a bipartite graph with at least one edge, then

$$\frac{2m\sqrt{2}}{\sqrt{n} + 2} \leq LEL(G)$$

(13)

with equality if and only if $G \cong K_2$.

4.2 Bounds involving vertex degrees

Further bounds are obtained by taking into consideration degrees of vertices, either the extremal vertex degrees only or the whole sequence of vertex degrees. Let $d_1 \geq \cdots \geq d_n$ denote the non-increasing sequence of vertex degrees. The largest and the smallest vertex degree will also be denoted as $\Delta = d_1$ and $\delta = d_n$.

Theorem 17. [40] If $G$ has $p$ connected components and at least one edge, then

$$LEL(G) \leq \sqrt{\Delta + 1} + \sqrt{(n - p - 1)(2m - \Delta - 1)}.$$ 

The previous bound was also obtained by Zhou [77] in the case of connected graphs ($p = 1$), in which case the equality is attained if and only if $G \cong K_n$ or $G \cong S_n$. 
Theorem 18. [8] If $G$ is a connected, non-complete graph with at least three vertices, then

$$\text{LEL}(G) \leq \sqrt{\Delta + 1} + \sqrt{\delta} + \sqrt{(n - 3)(2m - \Delta - \delta - 1)}$$

with equality if and only if $G \cong S_n$, $G \cong 2K_1 \lor K_{n-2}$ or $G \cong (K_1 \cup K_{n-2}) \lor K_1$.

Theorem 19. [6,43] If $G$ is a connected bipartite graph with at least three vertices, then

$$\text{LEL}(G) \leq \sqrt{\frac{\Delta + \delta + \sqrt{(\Delta - \delta)^2 + 4\Delta}}{2} + \left(n - 2\right)\left(2m - \frac{\Delta + \delta + \sqrt{(\Delta - \delta)^2 + 4\Delta}}{2}\right)}$$

with equality if and only if $G \cong S_n$.

The following bound improves (13) when $m > \Delta^2$.

Theorem 20. [73] If $G$ has $n$ vertices, $m$ edges and maximum vertex degree $\Delta$, then

$$\sqrt{\frac{8m^3}{n\Delta^2 + 2m}} \leq \text{LEL}(G)$$

(14)

with equality if and only if $G \cong K_n$.

The next two bounds rely on the fact that for $n \geq 2$ the sequence of vertex degrees is majorized by the sequence of Laplacian eigenvalues of a graph [17]. The first bound is proved in [40], and the second in [42].

Theorem 21. [40,42] If $d_1 \geq \cdots \geq d_n$ are degrees of vertices of $G$, then

$$\text{LEL}(G) \leq \sqrt{d_1 + 1} + \sqrt{d_2} + \cdots + \sqrt{d_{n-1}} + \sqrt{d_n - 1}$$

with equality if and only if $G \cong S_n$, and also

$$\text{LEL}(G) < \sqrt{d_1} + \sqrt{d_2} + \cdots + \sqrt{d_{n-1}} + \sqrt{d_n}.$$

Expressed in terms of the first Zagreb index $M_1 = \sum_{i=1}^{n} d_i^2$, the next two bounds also depend on the sequence of vertex degrees. Here, it is exploited that the sum of the squares of the Laplacian eigenvalues is $M_1 + 2m$, compare also the expressions for the Laplacian coefficients (4). The first theorem improves the bound (14) above.

Theorem 22. [22] For a graph $G$ with the first Zagreb index $M_1$ holds

$$\sqrt{\frac{8m^3}{M_1 + 2m}} \leq \text{LEL}(G)$$

with equality if and only if all nonzero Laplacian eigenvalues are equal.
The next bound appeared independently in three different papers.

**Theorem 23.** [43,78,80] If $G$ is a bipartite graph with at least one edge, then

$$LEL(G) \leq \sqrt{\frac{M_1}{m} + \sqrt{(n - 2) \left(2m - \frac{M_1}{m}\right)}}$$

with equality if and only if $G \cong S_n$, $G \cong K_{n/2,n/2}$ for even $n$, or $G \cong K_2 \cup (n - 2)K_1$.

### 4.3 Bounds involving the number of spanning trees

A number of bounds for the $LEL$ depend on the number of spanning trees $t_G$ in $G$, since the value $P$ in (8) is equal to $nt_G$. Although $t_G$ can be calculated using the Matrix-tree theorem [70] for general graphs, these bounds are most useful when applied to graphs with a few edges, that necessarily have a small number of spanning trees.

Gutman [20] improved the bound (9) by relying on $\mu_n = 0$ and Kober’s inequality [36] to obtain

**Theorem 24.** [20] If $G$ has $t$ spanning trees, then

$$LEL(G) \leq \sqrt{\frac{2m(n - 1)^2}{n} + (n - 1)(nt)^{\frac{1}{n - 1}}}.$$  

This bound was further improved by Das et al. [12].

**Theorem 25.** [12] If $G$ has $t$ spanning trees, then

$$LEL(G) \leq \sqrt{n} + \sqrt{(n - 3)(2m - \Delta - 1) + (n - 2) \left(\frac{nt}{\Delta + 1}\right)^{\frac{1}{n - 2}}}$$

with equality if and only if $G \cong K_n$ or $G \cong S_n$.

Gutman [20] improved the bound (11) as well, again using $\mu_n = 0$ and Kober’s inequality [36].

**Theorem 26.** [20] If $G$ has $t$ spanning trees, then

$$\sqrt{\frac{4m(n - 1)}{n} + (n - 1)(n - 2)(nt)^{\frac{1}{n - 1}}} \leq LEL(G).$$  

(15)

The next bound is incomparable to (15), despite its relatively similar form with respect to the term $(nt)^{\frac{1}{n - 1}}$. 

Theorem 27. [5, 12] If $G$ is a connected graph with at least three vertices and $t$ spanning trees, then

$$\sqrt{\left(1 + \frac{1}{n-2}\right) \left[(n-1)^2(\frac{1}{n-1} - 2m)\right]} \leq LEL(G)$$

with equality if and only if $G \cong K_n$.

Das et al. [12] improved the bound (15) to obtain

Theorem 28. [12] If $G$ has $t$ spanning trees, then

$$\sqrt{\Delta + 1} + \sqrt{2m - n + (n - 2)(n - 3)\frac{1}{n-1}} \leq LEL(G)$$

(16)

with equality if and only if $G \cong K_n$ or $G \cong S_n$.

The terms $\sqrt{\Delta + 1}$ and $nt$ are common terms in several further lower bounds.

Theorem 29. [77] If $G$ is a connected graph with $n \geq 3$ vertices and $t$ spanning trees, then

$$\sqrt{\Delta + 1} + (n - 2)\left(\frac{nt}{\Delta + 1}\right)^{\frac{1}{n-1}} \leq LEL(G)$$

with equality if and only if $G \cong K_n$ or $G \cong S_n$.

The previous bound was slightly improved by Das et al. [15].

Theorem 30. [15] If $G$ is a connected graph with $n \geq 3$ vertices and $t$ spanning trees, then

$$\sqrt{\Delta + 1} + (n - 2)\left(\frac{nt}{\Delta + 1}\right)^{\frac{1}{n-1}} + \left(\sqrt{d_2} - \sqrt{\delta}\right)^2 \leq LEL(G).$$

Another lower bound of a similar form was obtained in [12].

Theorem 31. [12] If $G$ is a connected graph with $n \geq 3$ vertices and $t$ spanning trees, then

$$\sqrt{\Delta + 1} + (n - 2)(nt)^{\frac{1}{n-1}}\left(\frac{2(nt)^{\frac{1}{n-1}}(n-2)}{\Delta + 1} - 1\right) \leq LEL(G)$$

with equality if and only if $G \cong K_n$.

The following bound is another example of a result that managed to appear independently in three different papers.
Theorem 32. [6,43,80] If $G$ is a bipartite graph with at least three vertices and $t$ spanning trees, then

$$\sqrt{\frac{M_1}{m}} + (n-2) \left(\frac{nmt}{M_1}\right)^{\frac{1}{n-4}} \leq \text{LEL}(G)$$

with equality if and only if $G \cong S_n$ or $G \cong K_{n/2,n/2}$.

A few of the bounds depend on the value $\beta = \frac{1}{2} \left(\Delta + \delta + \sqrt{(\Delta - \delta)^2 + 4\Delta}\right)$ that is slightly larger than the maximum vertex degree $\Delta$.

Theorem 33. [5,6] If $G$ is a connected bipartite graph with at least three vertices and $t$ spanning trees, then

$$\text{LEL}(G) \leq \sqrt{n} + \sqrt{(n-3)(2m-\beta) + (n-2) \left(\frac{nt}{\beta}\right)^{\frac{1}{n-4}}}$$

$$\sqrt{\beta} + (n-2) \left(\frac{nt}{\beta}\right)^{\frac{1}{n-4}} \leq \text{LEL}(G)$$

$$\sqrt{\beta} + \sqrt{2m-n + (n-2)(n-3)t^{\frac{1}{n-4}}} \leq \text{LEL}(G)$$

each with equality if and only if $G \cong S_n$.

4.4 Bounds involving other graph invariants

Restricting the value of certain graph invariants often imposes conditions on the structure of graphs that are sufficient for characterizing the extremal graphs. For example, Zhu [81] proved the following two bounds in terms of the vertex connectivity and the chromatic number of $G$ by using the monotonicity of the $\text{LEL}$ with respect to edge addition. Note that these two bounds independently follow from the results of Qiu and Yan [56] on Laplacian coefficients (see Theorem 11).

Theorem 34. [81] If the vertex connectivity of $G$ is at most $k$, then

$$\text{LEL}(G) \leq k\sqrt{n} + \sqrt{k} + (n-k-2)\sqrt{n-1}$$

with equality if and only if $G \cong K_k \bigvee (K_1 \cup K_{n-k-1})$.

Theorem 35. [81] Let $r = \lfloor n/\chi \rfloor$ and $s = n - r\chi$. Then

$$\text{LEL}(G) \leq (\chi - 1)\sqrt{n} + (\chi - s)(r-1)\sqrt{n-r} + sr\sqrt{n-r-1}$$

with equality if and only if $G \cong K_{r,r+1,...,r+1}$. 
Another bound that is easily obtained by monotonicity of the LEL with respect to edge addition is the following:

**Theorem 36.** [12] If $G$ has independence number $\alpha$, then

$$LEL(G) \leq (n - \alpha)\sqrt{n} + (\alpha - 1)\sqrt{n - \alpha}$$

with equality if and only if $G \cong K_{n-\alpha} \sqrt{K_\alpha}$.

The following bounds present an improvement upon the bound (12) in cases when it is known that the clique number of a graph is at most $r$.

**Theorem 37.** [22] Let $G$ be a $K_{r+1}$-free graph. Then

$$\frac{2m}{\sqrt{\frac{r-1}{r}n + 1}} \leq LEL(G)$$

with equality if and only if $G \cong K_r$ or $G \cong K_n$.

This bound also holds for the incidence energy of a graph. Another bound, which may be better if the number of edges of a graph $G$ is far less than that of a Turán graph [69], is as follows.

**Theorem 38.** [15] Let $G$ be a connected graph with clique number $\omega$. Then

$$\sqrt{\Delta + 1} + (\omega - 2)\sqrt{\omega} + \Delta - \omega + 1 \leq LEL(G)$$

with equality if and only if $G \cong K_\omega$ or $G$ is a graph obtained from $K_\omega$ by attaching $n - \omega$ pendant edges to one of its vertices.

In addition to the ordinary graph distance, Klein and Randić [35] conceived the resistance distance between a pair of vertices as the electrical resistance between them, under the assumption that the graph represents an electrical network in which the resistance between any two adjacent vertices is 1 Ohm. The Kirchhoff index, denoted as $Kf(G)$, is defined as the sum of resistance distances between all pairs of vertices of $G$. The Kirchhoff index has a very nice mathematical representation [23,82] as

$$Kf(G) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i}.$$ 

Although $LEL(K_n) = (n - 1)\sqrt{n} > n - 1 = Kf(K_n)$, Das et al. [14] showed that

$$LEL(G) < Kf(G)$$
whenever $G$ is a connected graph with $2m \leq (n-2)n^{2/3} + \delta$ or $2m \leq (n-1)n^{2/3}$. They also posed the problem of what would be the smallest constant $c$ (which may depend on the number of vertices $n$ or the maximum vertex degree $\Delta$), such that $Kf(G) < LEL(G)$ for each connected graph with at least $c$ edges.

Arsić et al. [3] showed that for any positive integer $c$ there exist only finitely many graphs $G$ with cyclomatic number $c$ satisfying $Kf(G) < LEL(G)$, and determined such graphs among those with cyclomatic number at most four. They also showed that $Kf(G) < LEL(G)$ holds whenever $G$ contains $K_{n/2} \square K_2$ as a spanning subgraph for even $n$, or contains $(K_{(n-1)/2} \square K_2) \mathbin{\lor} K_1$ as a spanning subgraph for odd $n$ ($\square$ denoting the Cartesian product).

Das and Mojallal [13] have recently found a bound involving both the Laplacian energy $LE(G)$ and the Laplacian-energy-like invariant $LEL(G)$.

\[
n^2 (LE(G) - n)^2 + 8mn \left(LEL(G) - \sqrt{n}\right)^2 \leq (4mn - 4m - n(\Delta + 1))^2
\]

with equality if and only if $G \cong K_n$ or $G \cong S_n$.

5 Various Results

In this final section, we discuss various interesting results on the $LEL$ that do not fit into any of the previous sections. Specifically, we look at Nordhaus–Gaddum type inequalities (concerning the $LEL$ of a graph and its complement), the behavior of the $LEL$ under certain graph operations (most notably the $LEL$ of line graphs), random graphs, and further special classes of graphs such as grid graphs.

5.1 Nordhaus–Gaddum type relations

In 1956, Nordhaus and Gaddum gave lower and upper bounds on the sum and the product of the chromatic number of a graph and its complement [51]. Since then, similar relations in which lower and upper bounds are given for the sum and the product of the values of a certain invariant for a graph and its complement are called Nordhaus–Gaddum type bounds. There are several hundred such bounds nowadays, and they are surveyed in detail in [2]. This survey, however, does not contain the following Nordhaus–Gaddum type relations for the $LEL$. 
Theorem 39. [26] If $G$ has at least two vertices, then

$$(n - 1)\sqrt{n} \leq \text{LEL}(G) + \text{LEL}(\overline{G}) \leq \sqrt{2(n + 1)} + \sqrt{2(n - 2)(n^2 - 2n - 1)}$$

with left equality if and only if $G \cong K_n$ or $G \cong \overline{K_n}$.

The left hand side also appears in [75], where the difference between $\text{LEL}(G)$ and $\text{LEL}(\overline{G})$ is studied as well (amongst other things, it is shown that the four-vertex path $P_4$ is the only tree $T$ such that $\text{LEL}(T) = \text{LEL}(\overline{T})$). Slightly better relations involving also the number of spanning trees were found by Das et al. [15].

Theorem 40. [15] Let $\overline{t}$ be the number of spanning trees in the complement $\overline{G}$. Then

$$(n - 1)n^{\frac{1}{n-2}} \left(\sqrt{\overline{t}} + \sqrt{\overline{t}}\right)^{\frac{1}{n-1}} \leq \text{LEL}(G) + \text{LEL}(\overline{G}) \leq \sqrt{n} + \sqrt{n - \Delta - 1} + (n - 2)\sqrt{2n}$$

with either equality if and only if $G \cong K_n$ or $G \cong \overline{K_n}$.

The following bound manages to avoid the factor $\sqrt{2}$ from the two previous upper bounds, provided that the graph has a connected complement and sufficiently many vertices.

Theorem 41. [8] If $G$ is a connected graph with at least five vertices and a connected complement $\overline{G}$, then

$$\text{LEL}(G) + \text{LEL}(\overline{G}) \leq (n - 1)\sqrt{n + \sqrt{n(n - 1)}}$$

with equality if and only if $G$ is a conference graph.

Finally, You and Lin [75] proved the following bound involving both the number of edges and the maximum degree:

Theorem 42. [75] For any graph $G$ with at least two vertices,

$$\text{LEL}(G) + \text{LEL}(\overline{G}) \leq \sqrt{1 + \Delta + \sqrt{n - 2\sqrt{2m - 1 - \Delta} + \sqrt{n - 1 - \Delta}}$$

$$\quad + \sqrt{n - 2\sqrt{n(n - 2) - 2m + 1 + \Delta}}$$

$$\leq \sqrt{n + \sqrt{n - 2(\sqrt{2m - n} + \sqrt{n(n - 1) - 2m})}$$

and equality holds for a star or a complete graph (for the second inequality, one also needs to assume that $2m \geq n$).
5.2 The LEL of derived graphs

Derived graphs are results of unary operations on other graphs. The most prominent operation is that of a line graph: the line graph $L(G)$ has as its vertex set the edge set of the graph $G$, with two vertices in $L(G)$ adjacent if the corresponding edges in $G$ share a vertex. Other common derived graphs are the subdivision graph and the total graph. The subdivision graph $S(G)$ is obtained by replacing each edge of $G$ with a two-edge path. The total graph $T(G)$ has as its vertex set the union of the sets of vertices and edges of $G$, with two vertices in $T(G)$ adjacent if and only if the corresponding objects are either adjacent or incident in $G$. In case the underlying graph is regular, the Laplacian spectrum of derived graphs may be easily related to that of the initial graph, which leads to the following bounds.

Theorem 43. [73] If $G$ is an $r$-regular graph, then:

\[ 0 < \text{LEL}(L(G)) - \frac{n(r - 2)\sqrt{2r}}{2} \leq \sqrt{n(n - 1)r} \]

with right equality if and only if $G \cong K_n$;

\[ (n - 1)\sqrt{r + 2} < \text{LEL}(S(G)) - \frac{n(r - 2)\sqrt{2}}{2} - \sqrt{r + 2} \leq (n - 1)(\sqrt{r} + \sqrt{2}) \]

with right equality if and only if $G \cong K_2$; and

\[ (n - 1)\sqrt{r + 2} < \text{LEL}(T(G)) - \frac{n(r - 2)\sqrt{2(r + 1)}}{2} - \sqrt{r + 2} \leq (n - 1)\sqrt{3r} \]

with right equality if and only if $G \cong K_2$.

Some further bounds on the LEL of graphs derived from regular and semiregular graphs are obtained in [55].

Wang [72] also studied iterated line graphs $L^k(G)$ of regular graphs; if $n_k$ denotes the order of $L^k(G)$ and $r_k$ the common degree of all vertices of $L^k(G)$, then the limit

\[ \lim_{k \to \infty} \frac{\text{LEL}(L^k(G))}{n_k \sqrt{r_{k-1}}} = \sqrt{2} \]

turns out to be independent of the graph $G$, which parallels a result on the energy of iterated line graphs.

To conclude this subsection, we mention a recent result of Pastén and Rojo: let $G$ be a graph with $r$ vertices that have $s_1, s_2, \ldots, s_r$ pendant vertices as neighbors. If these are
identified with the vertices of graphs $H_1, H_2, \ldots, H_r$ of order $s_1, s_2, \ldots, s_r$ respectively, we obtain a graph $G(H_1, H_2, \ldots, H_r)$. Surprisingly, it turns out that the difference

$$LEL(G(H_1, H_2, \ldots, H_r)) - LEL(G)$$

is independent of $G$ (it only depends on the graphs $H_1, H_2, \ldots, H_r$).

### 5.3 LEL-equienergetic graphs

Just like most other graph invariants, the LEL does not uniquely characterize a graph; evidently, if two graphs $G_1$ and $G_2$ have the same Laplacian spectrum, then $LEL(G_1) = LEL(G_2)$. Two non-isomorphic graphs $G_1$ and $G_2$ that satisfy this identity are generally called LEL-equienergetic. In comparison to other graph energies, finding equienergetic graphs that are not cospectral is comparatively difficult. For example, the complete graph $K_n$ and the graph $S_n + \{e\}$ obtained by adding an edge to a star have the same Laplacian energy while not being cospectral, as pointed out in [41]; however, there is no pair of connected non-cospectral graphs of order less than 8 that are LEL-equienergetic [41].

Three different pairs of connected non-cospectral LEL-equienergetic graphs were given in [41], as well as some pairs of almost LEL-equienergetic trees ("almost" meaning that the difference is less than $10^{-8}$). It was later proved by Liu and Liu [44] that there is in fact a pair of non-cospectral LEL-equienergetic connected graphs for every order $n \geq 12$, by giving an explicit construction: the graphs $H_1 = (K_3 \cup S_7 \cup (n-11)K_1) \vee K_1$ and $H_2 = (S_8 \cup S_3 \cup (n-12)K_1) \vee K_1$ form such a pair. Further similar constructions can be found in [44] as well. The question whether there exist non-cospectral LEL-equienergetic trees, however, still appears to be open.

More sophisticated arguments can be used to construct arbitrarily large sets of pairwise LEL-equienergetic graphs: so-called decomposable graphs (a graph is called decomposable if it can be obtained from isolated vertices by the operations of union and complement) were used in [61] to show that for every positive integer $k$, there exists a set of $k$ pairwise LEL-equienergetic graphs, and the number of vertices is at most linear in $k$. In [60], it was even shown that there are "many" equienergetic graphs for any order $n \geq 9$: in fact, there is always a set of at least $\left\lfloor \frac{n}{9} \right\rfloor + 1$ graphs of order $n$ with this property. This result uses the family of so-called threshold graphs: a threshold graph is constructed by recursively adding vertices in such a way that every new vertex is either
adjacent to all previous vertices or to none of them.

5.4 Random graphs

A paper by Du, Li and Li [16] investigates random graphs and provides very strong statements about the asymptotic behavior of the $LEL$ and other graph energies. Specifically, for the Erdős-Rényi random graph $G_{n,p}$, where the edge probability $p$ is constant, they obtained the following:

**Theorem 44.** [16] As $n \to \infty$,

$$LEL(G_{n,p}) = (\sqrt{p} + o(1)) n^{3/2}$$

holds asymptotically almost surely.

5.5 Further special classes of graphs

The result of the previous section shows that the “typical” order of $LEL$ is $n^{3/2}$. The asymptotic behavior was also studied for deterministic sequences of graphs, most notably perhaps for regular lattices [72]. For the square lattice $P_m \times P_n$ of dimensions $m \times n$, it was found that

$$LEL(P_m \times P_n) \sim 1.91618mn$$

as $m, n \to \infty$, the constant being given by $\pi^{-2} \int_0^\pi \int_0^\pi \sqrt{4 - 2 \cos x - 2 \cos y} \, dx \, dy$. This statement remains true for cylinders and tori, and similar results hold (with different constants) for triangular and hexagonal lattices. In order to obtain this asymptotic formula, one can exploit the fact that the Laplacian eigenvalues of grid graphs are known quite explicitly.

A Bethe tree of $k$ levels is a rooted tree whose root degree is $d$, all vertices from level 2 to level $k - 1$ have degree $d + 1$, and the vertices at the last level (level $k$) are leaves. Robbiano and Trevisan [58] obtained a recurrence for the $LEL$ of Bethe trees and used it to provide upper and lower bounds for it. Generalized forms of Bethe trees were studied in [7].
References


